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Artikel: AUTOMATIC GROUPS: A GUIDED TOUR
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EXAMPLES OF GROUPS THAT ARE NOT AUTOMATIC:

1. *Infinite torsion groups.* The mere existence of such groups is far from trivial, so it is not very disconcerting that such groups are not automatic (it is, perhaps, heartening). That infinite torsion groups are not automatic follows immediately from the well-known "pumping lemma" for finite state automata (see [Gi] for the proof).

2. *Nilpotent groups.* Finitely generated nilpotent groups which do not contain an abelian subgroup of finite index are not automatic. This was first proved by Holt. For example the three-dimensional Heisenberg group $H_3 = < a, b, c: [a, b] = c, [a, c] = 1 = [b, c] >$, the simplest non-abelian nilpotent group, has a cubic isoperimetric function (see property 7 below) and so is not automatic. The fact that nilpotent groups are not automatic is a bit surprising and annoying, considering the fact that nilpotent groups are quite common and have an easily solved word problem.

3. $SL_n(\mathbf{Z}), n \geqslant 3$. Note that $SL_2(\mathbf{Z})$ contains a free subgroup of index six, and so is automatic. The proof that $SL_n(\mathbf{Z}), n \geqslant 3$ is not automatic involves finding a contractible manifold on which $SL_n(\mathbf{Z})$ acts with compact quotient, and showing that a higher-dimensional isoperimetric inequality is not satisfied by that space. The search for this manifold involves the study of the symmetric space $SL_n(\mathbf{R}) / SO_n(\mathbf{R})$.

4. *Baumslag-Solitar Groups.* The group $G_{p,q} = < x, y: yx^p y^{-1} = x^q >$ is not automatic unless $p = 0$, $q = 0$ or $p = \pm q$. These groups provide examples of groups which are not automatic but are *asynchronously automatic* (see [BGSS, E *et al.*]).

5. HYPERBOLIC GROUPS ARE AUTOMATIC

It is most often the case that proving that a group G is automatic requires doing quite a bit of geometry in a space on which G acts in a geometric way. As an example we prove the result of Cannon that cocompact discrete groups of hyperbolic isometries are automatic; in fact we show this more generally for fundamental groups of compact manifolds with (not necessarily constant) strictly negative sectional curvatures.

A path $\alpha: [a, b] \to X$ in a metric space X is a *quasi-geodesic* if it is a geodesic up to constants; that is, there exists a K such that

$$1 / K(t_2 - t_1) - K < d_X(\alpha(t_1), \alpha(t_2)) < K(t_2 - t_1) + K$$

for all $t_1 < t_2$ in $[a, b]$; in this case α is called a *K-quasi-geodesic.* A quasi-geodesic is a 'geodesic in-the-large' (hence the adding and subtracting of the constant K). One of the most important facts about negatively curved spaces is that quasi-geodesics are close to geodesics.

LEMMA 2 (Morse-Mostow). *Let M be a compact manifold with strictly negative sectional curvatures. Then there exists a constant $C = C(K)$ such that any finite K-quasi-geodesic α in the universal cover $\tilde{M}$ lies in the C-neighborhood of the geodesic joining the endpoints of α.*

This lemma, implicit in a 1924 paper of Morse ([Mo]), was used in the proof of Mostow's rigidity theorem; a proof is given in [Th2]. Note that the situation is quite different for spaces which are not negatively curved; consider the logarithmic spiral in the euclidean plane.

If G is the fundamental group of a compact Riemannian manifold M, then a natural copy of the Cayley graph $\Gamma_S(G)$ sits inside of the universal cover $\tilde{M}$ of M; namely, choose a basepoint $m \in M$ and a lift $\tilde{m} \in \tilde{M}$ of m, put a vertex at $g(\tilde{m})$ for each deck transformation $g \in G$, and for each edge from g to $g \cdot s$ in the Cayley graph connect $g(\tilde{m})$ and $gs(\tilde{m})$ by a geodesic in $\tilde{M}$ (here $\tilde{M}$ is given the metric induced by that on M, so that the deck transformations act as isometries). It is a fundamental fact that paths in the Cayley graph $\Gamma_S(G)$ that are geodesic are actually quasi-geodesics in $\tilde{M}$. This ties the (quasi-)geometry of the fundamental group together with the (quasi-)geometry of the universal cover.

THEOREM 3. *If M is a compact manifold with strictly negative sectional curvatures, and if S is any generating set for $G = \pi_1(M)$, then the set of geodesic (shortest) words in $\Gamma_S(G)$ is a regular language, and is part of an automatic structure for G; in particular G is automatic.*

We follow the proof idea given in [Th1].

Proof. We shall prove that the set L of geodesics in $\Gamma_S(G)$ is a regular language satisfying the k-fellow traveller property for some k.

By the comment above there is some constant K such that geodesics $u, v \in L$ which represent elements of G at distance one apart in $\Gamma_S(G)$ are K-quasi-geodesics in $\tilde{M}$, so by Mostow's Lemma they lie in a C-neighborhood of geodesics u' and v' in $\tilde{M}$ with the same endpoints as u and v; this is the only place where the strictly negative curvature assumption is used. Now u' and v' form two sides of a triangle whose third side has length at most $2K$, by the equation on page 13. Since $\tilde{M}$ is a non-positively curved space, u' and

v' are within Hausdorff distance $2K$ from each other. Hence u and v are Hausdorff distance of at most $2K + C + C$ apart (see figure 6), and from this it follows easily that, since u and v are geodesics in $\Gamma_S(G)$, they are k-fellow travellers for some constant k, and this k depends only on the curvature bound on M, not on u and v.

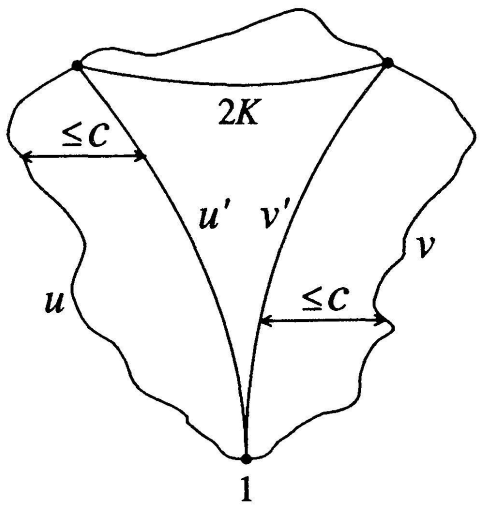

FIGURE 6

Why geodesics in $\Gamma_S(G)$ are fellow travellers

Notice that the subwords u_i and v_i consisting of the first i letters of u and v are geodesics, and $u_i v_i^{-1}$ lies in the k-ball B_k around the identity in $\Gamma_S(G)$.

We now build a finite state automaton W which recognizes precisely the set of geodesics in $\Gamma_S(G)$. As state set S of W we take the set of subsets of the k-ball B_k together with a fail state F, let the subset consisting of the identity be the start state, and take every state of S except for F as an accept state. Suppose the generator g of the word w is read when W is in the state T. Then W should go into the fail state if either T is the fail state or $g \in T$; otherwise W should go into the state

$$\{g^{-1}ta : t \in T, a \in S \cup S^{-1} \cup \{e\}\} \cap B_k$$

(see figure 7). The idea is to keep track of all paths which are competing with w for being the shortest path to $\bar{w}$; w is rejected as soon as a subword of w is longer than one of its competitors. The amount of information to remember is finite since we need to keep track only of word differences, all of which lie in a finite set (namely B_k). More precisely, after reading in the first i letters of w, the current state S_i (if it isn't the fail state) consists of precisely those elements t of B_k which satisfy the property that there is some path of length i from 1 to $w_i \cdot t$ which is a k-fellow traveller with w_i; this follows easily by induction on i.

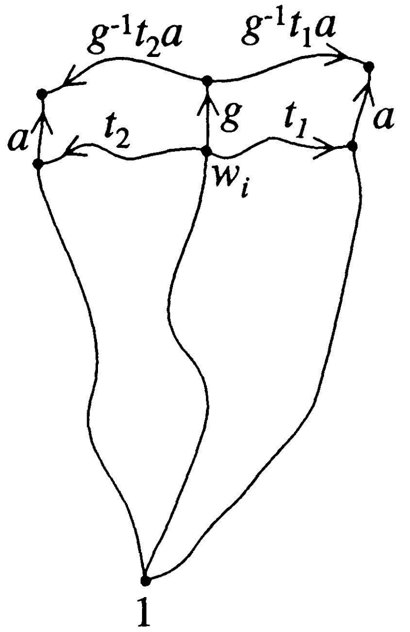

FIGURE 7

How to determine the new state after being in state T and reading the generator g
Here $t_1, t_2 \in T$, $a \in S \cup S^{-1}$

We claim that the finite state automaton W accepts precisely the set of geodesics. If the $(i+1)$'st generator of w lies in S_i, then w_{i+1} (hence w) is not a geodesic since $\overline{w_{i+1}}$ can be represented by a word of length i. Hence W accepts every geodesic word. Now suppose that w is not a geodesic; so there is some i such that w_i is a geodesic but there is some path v from 1 to $\overline{w_{i+1}}$ which is shorter than the path w_{i+1}; we may assume that v is geodesic. Since w_i and v are geodesics ending a distance one apart in the Cayley graph, they are k-fellow travellers. Note that v has length at most i, so by padding v we can make a path v' of length i which is a k-fellow traveller with w_i. But v' and $w_{i+1} = w_i \cdot g$ represent the same group element; hence $g \in S_i$. This shows that the fail state is entered upon reading the smallest initial subword of w which is not a geodesic; in particular the FSA W accepts only geodesic words. □

Cannon's original proof of theorem 3 is based on the notion of 'cone type' (see [Ca1, Ep1]). The idea is that geodesics in a hyperbolic group have only finitely many asymptotic behaviors (i.e. cone types), so building a geodesic generator by generator requires looking at a finite set (the set of cone types) and applying finitely many rules (determining the possible cone types after adding the next generator).