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THE CATEGORY OF NILMANIFOLDS

by John OPREA

ABSTRACT. The techniques of rational homotopy theory are used to compute the category of a nilmanifold: $\operatorname{cat}(M) = \dim M = \operatorname{rank}(\pi_1 M)$. This information is of interest to dynamicists since the theorem of Lusternik-Schnirelmann then shows that the number of critical points of a smooth function of M is bounded below by $\operatorname{rank}(\pi_1 M) + 1$.

INTRODUCTION

As a first step to understanding the structure of certain dynamical systems on nilmanifolds, one might hope to have computable lower bounds on the number of critical points of smooth functions. Of course, one is then led to the Lusternik-Schnirelmann definition of category and their well-known result that category $(+ 1)$ is such a bound. Unfortunately, category is rarely computable, so those who require numerical bounds often employ the fact that category majorizes cuplength. Hence cuplength (which, generally, is a more computable homotopy invariant than category) is the numerical invariant frequently sought for in order to provide a lower bound for the number of critical points of smooth functions on a manifold.

Indeed, some time ago, for the reasons above, Chris McCord asked me if I knew of a formula for the cuplength of a nilmanifold. I did not then, and after many computations I do not now! Thus, I pose:

QUESTION. What is the cuplength (with $\mathbf{Q}$-coefficients say) of a nilmanifold?

Suprisingly, however, the need for such knowledge by dynamicists is obviated by the following.

THEOREM 1. *If M is a (compact) nilmanifold, then* $\operatorname{cat}(M) = \dim(M) = \operatorname{rank}(\pi_1 M)$.

Hence, the best possible result which Lusternik-Schnirelmann theory can provide for nilmanifolds is the immediate.

COROLLARY. *The number of critical points of a smooth function on a (compact) nilmanifold M is bounded below by* $\operatorname{rank}(\pi_1 M) + 1$.

In fact, Theorem 1 was announced for all $K(\pi, 1)$'s by Eilenberg and Ganea [11]. Unfortunately, details of the proofs of their three fundamental propositions never appeared, thus contributing, I believe, to the ignorance of the result among the dynamicists and topologists of today. Indeed, this paper was originally written in response to Chris McCord's question and without knowledge of the Eilenberg-Ganea result. Furthermore, in looking at the Eilenberg-Ganea propositions, it is difficult to see the relationship between the structures of π and $K(\pi, 1)$ and the consequent determination of category as $\operatorname{rank}(\pi)$. I hope that the approach of this paper will remedy this defect, at least in the case of nilmanifolds. The beautiful structure theory of nilmanifolds (i.e. finitely generated torsionfree nilpotent groups) is ideally suited for an approach in terms of minimal models. In fact, in some sense, this paper is simply an exposition of just how well rational homotopy theory and nilmanifold theory fit together (in the representative situation of determining category).

Theorem 1 will be given a simple ("up to" the machinery of rational homotopy theory) proof in §4. Since this paper is written for workers in dynamical systems, I have tried to make it somewhat self-contained. Therefore, §1 and §2 are devoted to recollections on category and its rational homotopy description respectively. §3 recollects structural knowledge of nilmanifolds and §5 presents an analogue of Theorem 1 for iterated principal bundles. (The basic reference for the rational homotopy version of L.S. category is [3]; I have attempted to cull the essential ingredients for the proof of Theorem 1, but the reader will find other interesting applications in that work. Also see [2].)

§1. CATEGORY

The *category* of a space M, $\operatorname{cat}(M)$, is the least integer m so that M is covered by $m + 1$ open subsets each of which is contractible within M.

An equivalent definition (at least for the spaces we consider here) was given by G. Whitehead (see [10]): Let M^{m+1} denote the $(m + 1)$-fold product and