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**Artikel:** THE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION ON THE LINE  $\sigma = 1$   
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The error term in (2) is best possible, and for the range  $1/2 < \sigma < 3/4$  the above asymptotic formula has been considerably refined by K. Matsumoto [5] (see also [4]).

It seems that the case  $\sigma = 1$  in (1) has not received much attention in the literature. The aim of this paper is to discuss this problem. We shall prove the following

**THEOREM.** *Let for  $T > 1$  the function  $R(T)$  be defined by*

$$(3) \quad \int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T - \pi \log T + R(T).$$

Then

$$(4) \quad R(T) = O(\log^{2/3} T (\log \log T)^{1/3}),$$

$$(5) \quad \int_1^T R(t) dt = O(T),$$

and

$$(6) \quad \int_1^T R^2(t) dt = O(T(\log \log T)^4).$$

*Remarks.* In view of (4) it is seen that  $R(T)$  represents the error term in the mean square formula for  $\zeta(s)$  on the line  $\sigma = 1$ .

One often takes the lower limit of integration in (1) as zero. However, in our case ( $\sigma = 1$ ) this cannot be done, since  $\zeta(s)$  has a pole at  $s = 1$ . If in (3) we take some other positive number as the lower limit of integration, then obviously the value of  $R(T)$  will be changed by a constant only.

The method used in the proof of our theorem may be used to evaluate mean values of certain other zeta-functions on the line  $\sigma = 1$ . These will be dealt with elsewhere.

The upper bound in (4) contains information about the order of  $\zeta(1+iT)$ . Namely, from the general inequalities proved in [1] and [2] it may be deduced that

$$\zeta^k(1+iT) \ll \log^{2/3} T (\log \log T)^{1/3} \int_{T+\delta}^{T+\delta} |\zeta(1+it)|^k dt + 1$$

for any fixed integer  $k \geq 1$  and  $\delta = (\log \log T / \log T)^{2/3}$ . Hence we obtain

$$\zeta(1+iT) \ll \log^{2/3} T (\log \log T)^{1/3}.$$

This bound is close to the classical bound of I. M. Vinogradov (see Ch. 6 of [3])

$$\zeta(1 + iT) \ll \log^{2/3} T,$$

which for more than 30 years is the sharpest one, and follows from (19).

In view of (5) it seems plausible to conjecture that, as  $T \rightarrow \infty$ ,

$$\int_1^T R^2(t) dt \sim AT$$

for some  $A > 0$ .

*Acknowledgement.* We are very much indebted to the referee for his remarks concerning the proof of (4).

*Proof of the Theorem.* We start from the simplest approximate functional equation for  $\zeta(s)$  (see Th. 1.8 of [3]). Namely, for  $0 < \sigma_0 \leq \sigma \leq 2$ ,  $x \geq |t|/\pi$ ,  $s = \sigma + it$ , we have

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}).$$

For  $t \leq T$  this gives

$$(7) \quad \zeta(1 + it) = \sum_{n \leq T} n^{-1-it} + \frac{T^{-it}}{it} + O\left(\frac{1}{T}\right).$$

Since  $|z|^2 = z\bar{z}$  we obtain from (7)

$$(8) \quad \int_1^T |\zeta(1 + it)|^2 dt = \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right|^2 dt - 2\operatorname{Re} \left\{ \frac{1}{i} \int_1^T \sum_{n \leq T} \left(\frac{T}{n}\right)^{it} \frac{dt}{nt} \right\} \\ + O\left(\frac{1}{T} \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right| dt\right) + O\left(\frac{\log T}{T}\right).$$

The main terms in the asymptotic formula (3) come from the first integral on the right-hand side of (8). To see this one can use the well-known Montgomery-Vaughan theorem for mean values of Dirichlet polynomials (see [6], also [7] and Ch. 5 of [3]), which says that

$$(9) \quad \int_0^T \left| \sum_{n \leq N} a_n n^{-it} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n |a_n|^2\right)$$

for arbitrary complex numbers  $a_n$ ,  $T > 0$  and  $N \geq 1$ . Therefore by the Cauchy-Schwarz inequality and (9) we have

$$(10) \quad \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right| dt \leq T^{1/2} \left( \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right|^2 dt \right)^{1/2} = O(T).$$

We also have

$$(11) \quad \int_1^T \sum_{n \leq T} \left( \frac{T}{n} \right)^{it} \frac{dt}{nt} = O(\log \log T).$$

To prove (11), let  $2 \leq H \leq T/2$  be a parameter. Then

$$\begin{aligned} & \int_1^T \sum_{n \leq T} \left( \frac{T}{n} \right)^{it} \frac{dt}{nt} = \sum_{n \leq T(1-1/H)} n^{-1} \int_1^T \left( \frac{T}{n} \right)^{it} \frac{dt}{t} \\ & + \sum_{T(1-1/H) < n \leq T} n^{-1} \int_1^T \left( \frac{T}{n} \right)^{it} \frac{dt}{t} = \sum_{n \leq T(1-1/H)} \frac{1}{n} \left\{ \frac{(T/n)^{it}}{it \log(T/n)} \Bigg|_1^T \right. \\ & \quad \left. + \int_1^T \frac{(T/n)^{it} dt}{it^2 \log(T/n)} \right\} + O \left( \sum_{T(1-1/H) < n \leq T} \frac{\log T}{n} \right) \\ & = O \left( \sum_{n \leq T(1-1/H)} \frac{1}{n \log(T/n)} \right) + O \left( \frac{\log T}{H} \right) \\ & \ll \int_1^{T(1-1/H)} \frac{dx}{x \log(T/x)} + \frac{\log T}{H} + 1 = \int_{(1-1/H)^{-1}}^T \frac{du}{u \log u} + \frac{\log T}{H} + 1 \\ & = \log \log T - \log \log \left( 1 - \frac{1}{H} \right)^{-1} + \frac{\log T}{H} + 1 \ll \log \log T \end{aligned}$$

for  $H = \log T$ . Thus from (8), (10) and (11) it follows that

$$(12) \quad \int_1^T \left| \zeta(1+iu) \right|^2 du = \int_1^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du + O(\log \log T).$$

Further we have

$$(13) \quad \int_1^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du = \int_0^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du - \int_0^1 \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du$$

and observe that by the Euler-Maclaurin summation formula (or by (7)) we have, for  $0 < u \leq 1$ ,

$$\sum_{n \leq T} n^{-1-iu} = \int_1^T x^{-1-iu} dx + \gamma + O(u) + O\left(\frac{1}{T}\right) = \frac{1 - T^{-iu}}{iu} \\ + \gamma + O(u) + O\left(\frac{1}{T}\right),$$

where  $\gamma = 0.5772157\dots$  is Euler's constant. Therefore

$$\int_0^1 \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du = \int_0^1 \frac{|1 - T^{-iu}|^2}{u^2} du - 2\gamma \operatorname{Re} \left\{ i \int_0^1 \frac{1 - T^{-iu}}{u} du \right\} \\ (14) \quad + O(1) = 2 \int_0^1 \frac{1 - \cos(u \log T)}{u^2} du + 2\gamma \int_0^1 \frac{\sin(u \log T)}{u} du + O(1) \\ = 4 \int_0^1 \frac{\sin^2((u \log T)/2)}{u^2} du + O(1) = 2 \log T \int_0^{(\log T)/2} \left( \frac{\sin v}{v} \right)^2 dv + O(1) \\ = \pi \log T + O(1),$$

since

$$\int_0^{\infty} \left( \frac{\sin v}{v} \right)^2 dv = \frac{\pi}{2}.$$

In (12) – (14) we replace  $T$  by  $t$ , and suppose that  $T \leq t \leq 2T$ . From (7) we have

$$\sum_{T < n \leq t} n^{-1-it} = O\left(\frac{1}{T}\right).$$

Hence from the definition of  $R(T)$ , given by (3), it follows that for  $T \leq t \leq 2T$  we have

$$(15) \quad R(t) = \int_0^t \left( \left| \sum_{n \leq T} n^{-1-iu} \right|^2 - \zeta(2) \right) du + O(\log \log T).$$

This is the fundamental formula that will be used in the proof of (4) and (6). We start with the proof of (4), taking in (15)  $t = T$  and writing

$$\sum_{n \leq T} n^{-1-iu} = \sum_{n \leq N} + \sum_{N < n \leq 2N} + \sum_{2N < n \leq T} = \sum_1 + \sum_2 + \sum_3,$$

say, where

$$(16) \quad N = \exp(C \log^{2/3} T (\log \log T)^{1/3})$$

with a suitable constant  $C > 0$ . We have

$$(17) \quad \int_0^T \left| \sum_2 + \sum_3 \right|^2 du = \frac{1}{2} \int_{-T}^T \left| \sum_{N < n \leq T} n^{-1-iu} \right|^2 du \\ = T \sum_{N < n \leq T} n^{-2} + \sum_{N < m \neq n \leq T} \frac{(m/n)^{iT} - (m/n)^{-iT}}{2imn \log(m/n)}.$$

For the terms  $n < m \leq 2n$  in the second sum on the right-hand side of (17) we may put  $m = n + h$ , producing a sum of the form

$$(18) \quad \sum_{1 \leq h \leq T} \sum_{\max(N, h) < n \leq T} \frac{(1 + h/n)^{iT} - (1 + h/n)^{-iT}}{2i(n+h)n \log(1 + h/n)}.$$

To estimate the inner sum over  $n$  we shall apply the Vinogradov-Korobov estimate for exponential (zeta) sums. In its original form this says that

$$(19) \quad \sum_{M < m \leq M_1 \leq 2M} m^{it} \ll M \exp\left(-\frac{C \log^3 M}{\log^2 t}\right) \left(M_0 \leq M \leq \frac{1}{2}t, t \geq t_0\right),$$

and a proof (with  $C = 10^{-5}$ ) may be found in Ch. 6 of [3]. However, it is easily seen that the method of proof of (19) yields also

$$(20) \quad \sum_{M < m \leq M_1 \leq 2M} \left(1 + \frac{h}{m}\right)^{it} \ll M \exp\left(-\frac{C \log^3 M}{\log^2 t}\right) \left(M_0 \leq M_1 \leq \frac{1}{2}t, t \geq t_0\right)$$

with some absolute  $C > 0$ , provided that  $1 \leq h \leq m$ . Therefore (20) gives

$$\sum_{N' < n \leq N''} \left(1 + \frac{h}{n}\right)^{iT} \ll N' \log^{-4} T \quad (N \leq N' < N'' \leq 2N' \leq T),$$

provided that the constant  $C$  in (16) is sufficiently large. It follows by partial summation that the sum in (18) is  $\ll \log^{-2} T$ . Moreover the terms with  $m > 2n$  in (17) may be estimated directly by applying (19) to the sum over  $m$ , so that (17) becomes

$$(21) \quad \int_0^T \left| \sum_2 + \sum_3 \right|^2 du = T \sum_{N < n \leq T} n^{-2} + O(\log^{-2} T).$$

Similarly we find that

$$(22) \quad \int_{-T}^T \sum_{n \leq N} n^{-1-iu} \sum_{2N < m \leq T} m^{-1+iu} du \ll \log^{-2} T$$

for  $N$  given by (16) and  $C$  sufficiently large. Now by (15) we have

$$(23) \quad R(T) = \frac{1}{2} \int_{-T}^T \left| \sum_1 + \sum_2 + \sum_3 \right|^2 du - \zeta(2)T + O(\log \log T),$$

and we shall use the elementary identity

$$\begin{aligned} \left| \sum_1 + \sum_2 + \sum_3 \right|^2 &= \left| \sum_1 + \sum_2 \right|^2 - \left| \sum_2 \right|^2 + \left| \sum_2 + \sum_3 \right|^2 \\ &\quad + 2\operatorname{Re}(\sum_1 \bar{\sum}_3). \end{aligned}$$

By (9) the first two terms above contribute

$$T \sum_{n \leq N} n^{-2} + O(\log N)$$

to (23), while by (21) and (22) the contribution of the third and fourth term will be

$$T \sum_{N < n \leq T} n^{-2} + O(\log^{-2} T).$$

Therefore (23) gives

$$\begin{aligned} R(T) &= T \sum_{1 \leq n \leq T} n^{-2} - \zeta(2)T + O(\log N) + O(\log^{-2} T) \\ &= \zeta(2)T - \zeta(2)T + O(\log N) = O(\log^{2/3} T (\log \log T)^{1/3}), \end{aligned}$$

which proves (4). With a little more effort one could presumably remove the  $\log \log$  – factor in (4).

To prove (5) it suffices to prove

$$(24) \quad \int_T^{2T} R(t) dt = O(T),$$

and then to replace  $T$  by  $2^{-j}T$  and sum over  $j = 1, 2, \dots$ . For this purpose the error term in (15) is too large. Thus we use first (8), (10) and (14) to write, for  $T \leq t \leq 2T$ ,

$$\begin{aligned} R(t) &= \int_0^t \left( \left| \sum_{n \leq T} n^{-1-iu} \right|^2 - \zeta(2) \right) du - 2\operatorname{Re} \left\{ \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du}{inu} \right\} \\ &\quad + O(1), \end{aligned}$$

which is more precise than (15). We square out the first sum above and then integrate termwise, noting that interchanging  $m$  and  $n$  we have

$$\sum := \sum_{1 \leq m \neq n \leq T} \frac{1}{imn \log(m/n)} = \sum_{1 \leq m \neq n \leq T} \frac{1}{inm \log(n/m)} = -\sum,$$

hence  $\sum = 0$ . This gives, for  $T \leq t \leq 2T$ ,

$$(25) \quad R(t) = \sum_{1 \leq m \neq n \leq T} \frac{(m/n)^{it}}{imn \log(m/n)} - 2 \operatorname{Re} \left\{ \frac{1}{i} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du}{un} \right\} + O(1).$$

Then we integrate (25) to obtain

$$\int_T^{2T} R(t) dt = \sum_{1 \leq m \neq n \leq T} \frac{(m/n)^{iT} - (m/n)^{2iT}}{mn \log^2(m/n)} - 2 \operatorname{Re} \left\{ \frac{1}{i} \int_T^{2T} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \right\} + O(T).$$

But the modulus of the double sum above does not exceed

$$\begin{aligned} & \sum_{1 \leq n < m \leq T} \frac{4}{mn \log^2(m/n)} = \sum_{1 \leq n < m \leq T, n < m/2} + \sum_{1 \leq n < m \leq T, n \geq m/2} \\ & \ll \log^2 T + \sum_{n \leq T} \sum_{n < m \leq 2n} \frac{1}{m^2 \log^2(m/n)} \ll \log^2 T + \sum_{n \leq T} \sum_{n < m \leq 2n} \frac{1}{(m-n)^2} \\ & \ll T, \end{aligned}$$

and it remains to estimate

$$\begin{aligned} I &:= \int_T^{2T} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} = \int_T^{2T} \int_1^{\log T} \sum_{n \leq T/2} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \\ &+ \int_T^{2T} \int_{\log T}^t \sum_{n \leq T/2} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} + \int_T^{2T} \int_1^t \sum_{T/2 < n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. We have

$$\begin{aligned} I_1 &= \sum_{n \leq T/2} \frac{1}{n} \int_1^{\log T} \left( \int_T^{2T} \left( \frac{t}{n} \right)^{iu} \frac{du}{u} \right) dt \\ &= \sum_{n \leq T/2} \frac{1}{n} \int_1^{\log T} \left\{ \frac{(t/n)^{iu}}{iu \log(t/n)} \Big|_T^{2T} + \int_T^{2T} \frac{(t/n)^{iu}}{iu^2 \log(t/n)} du \right\} dt \\ &\ll \frac{\log T}{T} \sum_{n \leq T/2} \frac{1}{n \log(T/n)} \ll \frac{\log^2 T}{T}, \end{aligned}$$

$$\begin{aligned}
 I_2 &= \sum_{n \leq T/2} \frac{1}{n} \int_T^{2T} \left\{ \frac{(t/n)^{iu}}{iu \log(t/n)} \Big|_{\log T}^t + \int_{\log T}^t \frac{(t/n)^{iu}}{iu^2 \log(t/n)} du \right\} dt \\
 &\ll \sum_{n \leq T/2} \frac{T}{n \log(T/n) \log T} \ll T, \\
 I_3 &= \sum_{T/2 < n \leq T} \frac{1}{n} \int_1^{2T} \left( \int_{\max(u, T)}^{2T} \left(\frac{t}{n}\right)^{iu} dt \right) \frac{du}{u} \\
 &= \sum_{T/2 < n \leq T} \int_1^{2T} n^{-1-iu} \frac{(2T)^{iu+1} - (\max(u, T))^{iu+1}}{u(iu+1)} du \\
 &\ll T \sum_{T/2 < n \leq T} \frac{1}{n} \ll T,
 \end{aligned}$$

and (24) follows.

It remains yet to prove (6), which follows from

$$(26) \quad \int_T^{2T} R^2(t) dt = O(T(\log \log T)^4).$$

We use (25) and (11) to obtain

$$\begin{aligned}
 R^2(t) &\leq \sum_{a \neq b \leq T, c \neq d \leq T} \frac{-2 \left(\frac{ad}{bc}\right)^{it}}{abcd \log(a/b) \log(d/c)} + O((\log \log T)^2) \\
 &= \sum_{ad=bc} + \sum_{0 < |\log(ad/bc)| \leq T^{-1} \log^4 T} + \sum_{|\log(ad/bc)| > T^{-1} \log^4 T} \\
 &\quad + O((\log \log T)^2) = S_0 + S_1 + S_2 + O((\log \log T)^2),
 \end{aligned}$$

say. On integrating we obtain

$$\begin{aligned}
 &\int_T^{2T} S_2 dt \\
 &= 2 \sum_{a \neq b \leq T, c \neq d \leq T, |\log(ad/bc)| > T^{-1} \log^4 T} \frac{\left(\frac{ad}{bc}\right)^{iT} - \left(\frac{ad}{bc}\right)^{2iT}}{iabcd \log\left(\frac{a}{b}\right) \log\left(\frac{d}{c}\right) \log\left(\frac{ad}{bc}\right)} \\
 &\ll \frac{T}{\log^4 T} \left( \sum_{a \neq b \leq T} \frac{1}{ab |\log(a/b)|} \right)^2 \ll T,
 \end{aligned}$$

since an elementary argument easily gives

$$\sum_{a \neq b \leq T} \frac{1}{ab |\log(a/b)|} \ll \log^2 T.$$

The remaining sums  $S_0$  and  $S_1$  are not integrated, but it is sufficient to show that

$$(27) \quad S_0 \ll 1, S_1 \ll (\log \log T)^4.$$

We have

$$\begin{aligned} S_0 &= \sum_{ad=bc, a \neq b \leq T, c \neq d \leq T} \frac{1}{abcd \log(a/b) \log(d/c)} \\ &= \sum_k \frac{1}{k^2} \sum_{ad=bc=k, a \neq b \leq T} \frac{1}{\log^2(a/b)} = \sum_{a \neq b \leq T} \frac{1}{\log^2(a/b)} \sum_{a|k, b|k} \frac{1}{k^2} \\ &\ll \sum_{a \neq b \leq T} \frac{1}{[a, b]^2 \log^2(a/b)} \leq \sum_{j=1}^{\infty} \sum_{a \neq b, (a, b)=j} \frac{1}{\log^2(a/b)} \left(\frac{j}{ab}\right)^2 \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{a' \geq 1, b' \geq 1, a' \neq b'} \frac{1}{\log^2(a'/b') (a'b')^2} = O(1). \end{aligned}$$

The proof of the other bound in (27) is also elementary, but somewhat more involved. Write

$$S_1 = S_3 + S_4,$$

where in  $S_3$  summation is over  $a, b, c, d$  such that  $1 \leq a \neq b \leq T$ ,

$$1 \leq c \neq d \leq T, |\log a/b| \geq T^{-1/4}, |\log c/d| \geq T^{-1/4},$$

$$0 < \left| \log \frac{ad}{bc} \right| \leq T^{-1} \log^4 T,$$

and in  $S_4$  over the remaining values of  $a, b, c, d$ . Thus

$$\begin{aligned} S_3 &\ll T^{1/2} \sum_{a, b, c, d \leq T; |\log(ad/bc)| \leq T^{-1} \log^4 T} \frac{1}{abcd} \\ &\ll T^{1/2} \sum_{|\log(k/l)| \leq T^{-1} \log^4 T; k \neq l \leq T^2} \frac{d(k)d(l)}{kl}. \end{aligned}$$

Now observe that if  $k \neq l \geq 1$  are integers, then

$$(28) \quad \left| \log \frac{k}{l} \right| \geq \log \frac{l+1}{l} \geq \frac{1}{2l},$$

so that

$$l \geq \frac{1}{2} T \log^{-4} T, \quad \frac{|k-l|}{l} \ll \left| \log \left( 1 + \frac{k-l}{l} \right) \right| \ll T^{-1} \log^4 T.$$

Thus

$$S_3 \ll T^{1/2+\varepsilon} \sum_{T/(2\log^4 T) \leq l \leq T^2} \frac{1}{l^2} \left( \frac{\log^4 T}{T} + 1 \right) \ll T^{-1/2+2\varepsilon} \ll 1$$

for  $0 < \varepsilon \leq 1/4$ . In  $S_4$  we have either  $|\log(a/b)| \leq T^{-1/4}$  or  $|\log(d/c)| \leq T^{-1/4}$ . If the former holds, then

$$\left| \log \frac{d}{c} \right| = \left| \log \frac{ad}{bc} - \log \frac{a}{b} \right| \leq 2T^{-1/4},$$

so that in  $S_4$  we have  $|\log(a/b)| \leq 2T^{-1/4}$ ,  $|\log(d/c)| \leq 2T^{-1/4}$ , and also  $a \ll b \ll a, c \ll d \ll c$ . Setting  $a = b + j_1, c = d + j_2$ , we have that  $j_1 \ll bT^{-1/4}, j_2 \ll dT^{-1/4}$ , and

$$(29) \quad \left| \frac{j_1}{b} - \frac{j_2}{d} \right| \ll \frac{\log^4 T}{T},$$

since

$$\begin{aligned} \left| \frac{a}{b} - \frac{c}{d} \right| &= \frac{|ad - bc|}{bd} \ll \frac{|ad - bc|}{bc} \ll \left| \log \left( 1 + \frac{ad - bc}{bc} \right) \right| \\ &= \left| \log \frac{ad}{bc} \right| \ll \frac{\log^4 T}{T}. \end{aligned}$$

Hence

$$S_4 \ll \sum_{b,d,j_1,j_2}^* \frac{1}{bdj_1j_2},$$

where  $\sum^*$  denotes summation with the conditions  $b, d \leq T; j_1 \ll bT^{-1/4}, j_2 \ll dT^{-1/4}$  and (29) satisfied. We have

$$S_4 = S_5 + S_6,$$

where trivially

$$S_5 = \sum_{T \log^{-10} T \leq b, d, \leq T; j_1, j_2 \leq \log^{20} T} \frac{1}{bdj_1j_2} \ll (\log \log T)^4,$$

and in  $S_6$  we have (by symmetry) either

$$\text{i) } j_2 \geq \log^{20} T$$

or

$$\text{ii) } d \leq T \log^{-10} T.$$

From (29) it follows that

$$(30) \quad d - \frac{bj_2}{j_1} \ll \frac{bd \log^4 T}{j_1 T}.$$

Suppose that i) holds. Then  $d - bj_2 j_1^{-1} \ll j_1^{-1} b \log^4 T$  from (30), hence the corresponding part of  $S_6$  is

$$\begin{aligned} &\ll \sum_{b \leq T} \frac{1}{b} \sum_{j_1 \leq bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geq \log^{20} T} \frac{1}{j_2} \sum_{|d - bj_2 j_1^{-1}| \leq Cbj_1^{-1} \log^4 T} \frac{1}{d} \\ &\ll \sum_{b \leq T} \frac{1}{b} \sum_{j_1 \leq bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geq \log^{20} T} j_2^{-2} \log^4 T \ll \log^{-4} T. \end{aligned}$$

If ii) holds, then (30) gives

$$d - \frac{bj_2}{j_1} \ll \frac{b}{j_1} \log^{-6} T,$$

and the corresponding contribution to  $S_6$  will be again  $\ll \log^{-4} T$ . This proves (27) and completes the proof of the Theorem.