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covers of  $\Sigma - K$  and  $\bar{\Sigma} - \bar{K}$  respectively. Let  $\Delta_K(t) = \prod_{i>0} [H_i(X)]^{(-1)^{i+1}}$  and  $\Delta_{\bar{K}}(t) = \prod_{i>0} [H_i(\bar{X})]^{(-1)^{i+1}}$ . The Wang sequence shows that multiplication by  $t - 1$  induces an isomorphism on  $H_i(X)$  for  $i > 0$ , so that if we take the polynomial represented by  $[H_i(X)]$  and plug in  $t = 1$  we get  $\pm 1$ . (Indeed if we consider the ring homomorphism  $\varphi: \mathbf{Z}[t, t^{-1}] \rightarrow \mathbf{Z}$  defined by  $\varphi(t) = 1$ , then  $\varphi([H_i(X)])$  is a divisor of  $[H_i(X) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$ .) Thus  $[H_i(X)]$  represents a non-zero element in  $\mathbf{F}_p[t, t^{-1}]$ , and hence  $\Delta_K(t)$  and  $\Delta_{\bar{K}}(t)$  give well-defined elements of  $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$ . Then the considerations of §1 show:

**THEOREM 2.3.** *Let  $K$  be a  $G$ -periodic knot in a homology  $q$ -sphere  $\Sigma$  with fixed set  $B$ , where  $G$  is a group of prime power order  $p^r$ . Let  $\lambda$  be the linking number of  $K$  and  $B$ . Then*

$$\Delta_K(t) \doteq \Delta_{\bar{K}}(t)^{p^r} (1 + t + \dots + t^{\lambda-1})^{p^r-1} \pmod{p}.$$

### §3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any  $\lambda \equiv \pm 1 \pmod{8}$ , T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number  $\lambda$  in a homology 3-sphere  $\Omega$  whose  $C_2 \times C_2$ -cover branched over the link is a homology 3-sphere  $\Sigma$ . We will show that this congruence condition is necessary. Equivalently, we show

**THEOREM 3.1.** *Suppose the Klein 4-group  $G \times H \cong C_2 \times C_2$  acts on a homology 3-sphere  $\Sigma$  so that the fixed sets  $\Sigma^G$  and  $\Sigma^H$  are disjoint circles. Then their linking number  $\lambda$  is congruent to  $\pm 1$  modulo 8.*

*Proof.* We have

$$\begin{array}{ccc} \Sigma & \rightarrow & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \rightarrow & \Sigma/(G \times H). \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let  $K = \Sigma^G/G \subset \Sigma/G$  and  $\bar{K} = K/H \subset \Sigma/(G \times H)$ . Then  $K$  is a knot of period 2. Renormalize  $\Delta_K(t)$  and  $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$  so that  $\Delta_K(t) = \Delta_K(t^{-1})$ ,  $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$ , and  $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$ . Murasugi's congruence shows

$$(**) \quad \Delta_K(t) = \Delta_{\bar{K}}(t)^2(t^{(1-\lambda)/2} + \dots + 1 + \dots + t^{(\lambda-1)/2}) + 2f(t),$$

where  $f(t) \in \mathbf{Z}[t, t^{-1}]$  satisfies  $f(t) = f(t^{-1})$ . Writing

$$f(t) = a_n t^{-n} + \dots + a_0 + \dots + a_n t^n,$$

we see  $f(1) \equiv f(-1) \pmod{4}$ . Since  $\Sigma \rightarrow \Sigma/G$  is a 2-fold cover branched over  $K$ ,  $|\Delta_K(-1)| = |H_1(\Sigma)| = 1$ . So  $1 = \Delta_K(1) \equiv \Delta_K(-1) \pmod{4}$ , and we see  $\Delta_K(-1) = 1$ . Take equation  $(**)$  and plug in  $t = 1$  and  $t = -1$ :

$$\begin{aligned} 1 &= 1 \cdot \lambda + 2 \cdot f(1) \\ 1 &= 1 \cdot (-1)^{(\lambda-1)/2} + 2 \cdot f(-1). \end{aligned}$$

Thus  $\lambda \equiv (-1)^{(\lambda-1)/2} \pmod{8}$  so  $\lambda \equiv \pm 1 \pmod{8}$ .

Applying the high-dimensional version of Murasugi's congruence ones sees that if  $G \times H \cong C_2 \times C_2$  acts on a homology  $q$ -sphere  $\Sigma$  so that  $\Sigma^G$  is a homology  $q-2$  sphere and  $\Sigma^H$  is a circle disjoint from  $\Sigma^G$ , then their linking number  $\lambda$  is congruent to  $\pm 1$  modulo 8. This and considerations from  $L$ -theory lead us to conjecture that if  $G \times H \cong C_2 \times C_2$  acts on a homology  $q$ -sphere  $\Sigma$  so that  $\Sigma^G$  is a homology  $k$ -sphere and  $\Sigma^H$  is a homology  $q-k-1$ -sphere disjoint from  $\Sigma^G$ , then their linking number  $\lambda$  is congruent to  $\pm 1$  modulo 8.

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