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Autor: Davis, James F. / Livingston, Charles
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$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$

$$\delta = 1 - g$$

be elements of the group ring $\mathbb{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbb{F}_p[t, t^{-1}]$ -modules (all homology is with \mathbb{F}_p -coefficients).

$$\begin{aligned} 0 &\rightarrow C_*(X^G) \rightarrow C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \delta C_*(X) \oplus C_*(X^G) \rightarrow C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \rightarrow 0 \\ &\vdots \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \rightarrow 0. \end{aligned}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbb{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbb{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X)))$. The above homological considerations show

$$\begin{aligned} \chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\ \chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\ \chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\ &\vdots \\ \chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X). \end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p.$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1}.$$

Finally suppose G has order p^r . Let G_1 be a normal subgroup of index p . By the exact sequences above $\text{rk } H_*(X/G_1; \mathbb{F}_p) < \infty$. By applying inductively the result for the G_1 -action on X and the G/G_1 action on X/G_1 , Theorem B follows.

§2. HIGH-DIMENSIONAL PERIODIC KNOTS

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots

were introduced in the thesis of *R. Cruz* [C]. He showed that if there is a semifree \mathbf{Z}/q -action on S^n with non-empty fixed set and an invariant knot K^{n-2} disjoint from the fixed set, then the fixed set is S^1 if $q \neq 2$, and is S^1 or S^0 if $q = 2$.

For our purposes a knot K in a homology n -sphere Σ is an embedded $(n-2)$ -dimensional homology sphere. Let G be a finite group. The knot K is *G-periodic* if it is invariant under a semifree G -action on Σ with fixed set $B \cong S^1$ disjoint from K . To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient $\bar{\Sigma} = \Sigma/G$ will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

PROPOSITION 2.1. $H_*(\bar{\Sigma} - \bar{K}) \cong H_*(S^1)$.

First we need a lemma.

LEMMA 2.2. *The linking number $\lambda = \text{lk}(B, K)$ is relatively prime to the order of G .*

Proof. (See also [C, 2.1.1]). By restricting the action to a subgroup \mathbf{Z}/p of G , we will assume $G = \mathbf{Z}/p$, and show $(\lambda, p) = 1$. By applying the Lefschetz Fixed-Point Theorem to a generator g of \mathbf{Z}/p , we see that if n is odd, the action on K is orientation-preserving, while if n is even, then $p = 2$ and the action is orientation-reversing. For local coefficients we will use \mathbf{Z}^t , the integers with the $\mathbf{Z}[\mathbf{Z}/p]$ -module structure given by $(\sum a_i g^i) \cdot k = \sum a_i (-1)^{i(n+1)} k$.

Let $\bar{\Sigma} - B \rightarrow K(\mathbf{Z}/p, 1)$ classify the G -cover. We will consider the commutative diagram:

$$\begin{array}{ccccc}
 H_{n-2}(K; \mathbf{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K}; \mathbf{Z}^t) & \rightarrow & H_{n-2}(K(\mathbf{Z}/p, 1); \mathbf{Z}^t) \\
 (*) \quad \downarrow & & \downarrow & & \parallel \\
 H_{n-2}(\Sigma - B; \mathbf{Z}) & \rightarrow & H_{n-2}(\bar{\Sigma} - B; \mathbf{Z}^t) & \rightarrow & H_{n-2}(K(\mathbf{Z}/p, 1); \mathbf{Z}^t) .
 \end{array}$$

The two groups on the left are infinite cyclic and the left vertical map is multiplication by λ . A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}/p \rightarrow 0$.

The map α is isomorphic to $\mathbf{Z} \xrightarrow{\times p} \mathbf{Z}$ because it comes from a p -fold cover of $(n-2)$ -dimensional closed manifolds. The map

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) \rightarrow H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$$

we compute algebraically by using a free $\mathbf{Z}G$ -resolution of \mathbf{Z} as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on \bar{K} to K ,

$$C_*(K) = \{C_{n-2} \rightarrow \dots \rightarrow C_0\}$$

with the i -chains C_i free $\mathbf{Z}G$ -modules. By mapping a free $\mathbf{Z}G$ -module onto $\ker(C_{n-2} \rightarrow C_{n-3})$ and continuing inductively, one constructs a free $\mathbf{Z}G$ -resolution of \mathbf{Z}

$$D_* = \{\dots \rightarrow D_n \rightarrow D_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0\}.$$

It follows that

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) = H_{n-2}(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}^t)$$

maps onto $H_{n-2}(D_* \otimes_{\mathbf{Z}G} \mathbf{Z}^t) = H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$. Furthermore by using the standard $\mathbf{Z}G$ -resolution of \mathbf{Z} (see e.g. [Mac]), one easily computes that $H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t) \cong \mathbf{Z}/p$.

Choose a G -invariant normal disk to B in Σ and let S^{n-2} be its boundary. Then the inclusion $S^{n-2} \rightarrow \Sigma - B$ is a homology equivalence. By the comparison theorem applied to the spectral sequence of the G -coverings (see [Mac]), the bottom row of (*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbf{Z}) \rightarrow H_{n-2}(S^{n-2}/G; \mathbf{Z}^t) \rightarrow H_{n-2}(G; \mathbf{Z}^t),$$

and hence by the previous paragraph to $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}/p \rightarrow 0$. Thus $(\lambda, p) = 1$.

Proof of 2.1. Let N be an equivariant tubular neighborhood of B . Then

$$0 = H_*(\Sigma - K, N; \mathbf{Z}[1/\lambda]) = H_*(\Sigma - K - B, N - B; \mathbf{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$\begin{aligned} 0 &= H_*((\Sigma - K - B)/G, (N - B)/G; \mathbf{Z}[1/\lambda]) = H_*((\Sigma - K)/G, N/G; \mathbf{Z}[1/\lambda]) \\ &= H_*((\Sigma - K)/G, B; \mathbf{Z}[1/\lambda]), \end{aligned}$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence $B \rightarrow N/G$. Thus $H_*(\bar{\Sigma} - \bar{K})$ looks like $H_*(S^1)$ except possibly for some λ -torsion. But by 2.1, λ is prime to the order of G , so for all primes q dividing λ , the transfer map $\text{tr}: H_*(\bar{\Sigma} - \bar{K}; \mathbf{Z}/q) \rightarrow H_*(\Sigma - K; \mathbf{Z}/q)$ is injective so there is no extra λ -torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let X and \bar{X} be the infinite cyclic

covers of $\Sigma - K$ and $\bar{\Sigma} - \bar{K}$ respectively. Let $\Delta_K(t) = \prod_{i>0} [H_i(X)]^{(-1)^{i+1}}$ and $\Delta_{\bar{K}}(t) = \prod_{i>0} [H_i(\bar{X})]^{(-1)^{i+1}}$. The Wang sequence shows that multiplication by $t - 1$ induces an isomorphism on $H_i(X)$ for $i > 0$, so that if we take the polynomial represented by $[H_i(X)]$ and plug in $t = 1$ we get ± 1 . (Indeed if we consider the ring homomorphism $\phi: \mathbf{Z}[t, t^{-1}] \rightarrow \mathbf{Z}$ defined by $\phi(t) = 1$, then $\phi([H_i(X)])$ is a divisor of $[H_i(X) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$.) Thus $[H_i(X)]$ represented a non-zero element in $\mathbf{F}_p[t, t^{-1}]$, and hence $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ give well-defined elements of $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$. Then the considerations of §1 show:

THEOREM 2.3. *Let K be a G -periodic knot in a homology q -sphere Σ with fixed set B , where G is a group of prime power order p^r . Let λ be the linking number of K and B . Then*

$$\Delta_K(t) \equiv \Delta_{\bar{K}}(t)^{p^r} (1 + t + \dots + t^{\lambda-1})^{p^r-1} \pmod{p}.$$

§3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any $\lambda \equiv \pm 1 \pmod{8}$, T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number λ in a homology 3-sphere Ω whose $C_2 \times C_2$ -cover branched over the link is a homology 3-sphere Σ . We will show that this congruence condition is necessary. Equivalently, we show

THEOREM 3.1. *Suppose the Klein 4-group $G \times H \cong C_2 \times C_2$ acts on a homology 3-sphere Σ so that the fixed sets Σ^G and Σ^H are disjoint circles. Then their linking number λ is congruent to ± 1 modulo 8.*

Proof. We have

$$\begin{array}{ccc} \Sigma & \rightarrow & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \rightarrow & \Sigma/(G \times H). \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let $K = \Sigma^G/G \subset \Sigma/G$ and $\bar{K} = K/H \subset \Sigma/(G \times H)$. Then K is a knot of period 2. Renormalize $\Delta_K(t)$ and $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$ so that $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$, and $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$. Murasugi's congruence shows