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$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$
$$\delta = 1 - g$$

be elements of the group ring $\mathbf{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbf{F}_p[t, t^{-1}]$-modules (all homology is with $\mathbf{F}_p$-coefficients).

$$\begin{array}{l} 0 \to C_*(X^G) \to C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \to 0 \\ 0 \to \delta C_*(X) \oplus C_*(X^G) \to C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \to 0 \\ 0 \to \sigma C_*(X) \to \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \to 0 \\ \vdots \\ 0 \to \sigma C_*(X) \to \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \to 0 . \end{array}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbf{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbf{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X)))$. The above homological considerations show

$$\begin{aligned} \chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\ \chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\ \chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\ &\vdots \\ \chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X) . \end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p .$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1} .$$

Finally suppose G has order p^r. Let G_1 be a normal subgroup of index p. By the exact sequences above $\operatorname{rk} H_*(X/G_1; \mathbf{F}_p) < \infty$. By applying inductively the result for the G_1-action on X and the G/G_1 action on X/G_1, Theorem B follows.

§2. High-dimensional periodic knots

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots were introduced in the thesis of *R*. Cruz [C]. He showed that if there is a semifree $\mathbf{Z}/q$-action on S^n with non-empty fixed set and an invariant knot K^{n-2} disjoint from the fixed set, then the fixed set is S^1 if $q \neq 2$, and is S^1 or S^0 if $q = 2$.

For our purposes a knot K in a homology n-sphere Σ is an embedded $(n-2)$-dimensional homology sphere. Let G be a finite group. The knot K is *G-periodic* if it is invariant under a semifree G-action on Σ with fixed set $B \cong S^1$ disjoint from K. To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient $\bar{\Sigma} = \Sigma/G$ will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

PROPOSITION 2.1. $H_*(\bar{\Sigma} - \bar{K}) \cong H_*(S^1)$.

First we need a lemma.

LEMMA 2.2. *The linking number* $\lambda = \mathrm{lk}(B, K)$ *is relatively prime to the order of* G.

Proof. (See also [C, 2.1.1]). By restricting the action to a subgroup $\mathbf{Z}/p$ of G, we will assume $G = \mathbf{Z}/p$, and show $(\lambda, p) = 1$. By applying the Lefschetz Fixed-Point Theorem to a generator g of $\mathbf{Z}/p$, we see that if n is odd, the action on K is orientation-preserving, while if n is even, then $p = 2$ and the action is orientation-reversing. For local coefficients we will use $\mathbf{Z}^t$, the integers with the $\mathbf{Z}[\mathbf{Z}/p]$-module structure given by $(\Sigma a_i g^i) \cdot k = \Sigma a_i (-1)^{i(n+1)} k$.

Let $\bar{\Sigma} - B \to K(\mathbf{Z}/p, 1)$ classify the G-cover. We will consider the commutative diagram:

$$
(*)\qquad \begin{array}{ccccc}
H_{n-2}(K;\mathbf{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K};\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t) \\
\downarrow & & \downarrow & & \| \\
H_{n-2}(\Sigma - B;\mathbf{Z}) & \to & H_{n-2}(\bar{\Sigma} - B;\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t)\ .
\end{array}
$$

The two groups on the left are infinite cyclic and the left vertical map is multiplication by λ. A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$.

The map α is isomorphic to $\mathbf{Z} \xrightarrow{\times p} \mathbf{Z}$ because it comes from a p-fold cover of $(n-2)$-dimensional closed manifolds. The map

$$H_{n-2}(\bar{K};\mathbf{Z}^t) \to H_{n-2}(\mathbf{Z}/p;\mathbf{Z}^t)$$

we compute algebraically by using a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$ as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on $\bar{K}$ to K,

$$C_*(K) = \{C_{n-2} \to \dots \to C_0\}$$

with the i-chains C_i free $\mathbf{Z}G$-modules. By mapping a free $\mathbf{Z}G$-module onto $\ker(C_{n-2} \to C_{n-3})$ and continuing inductively, one constructs a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$

$$D_* = \{\dots \to D_n \to D_{n-1} \to C_{n-2} \to \dots \to C_0\} .$$

It follows that

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) = H_{n-2}(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}^t)$$

maps onto $H_{n-2}(D_* \otimes_{\mathbf{Z}G} \mathbf{Z}^t) = H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$. Furthermore by using the standard $\mathbf{Z}G$-resolution of $\mathbf{Z}$ (see e.g. [Mac]), one easily computes that $H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t) \cong \mathbf{Z}/p$.

Choose a G-invariant normal disk to B in Σ and let S^{n-2} be its boundary. Then the inclusion $S^{n-2} \to \Sigma - B$ is a homology equivalence. By the comparison theorem applied to the spectral sequence of the G-coverings (see [Mac]), the bottom row of (*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbf{Z}) \to H_{n-2}(S^{n-2}/G; \mathbf{Z}^t) \to H_{n-2}(G; \mathbf{Z}^t) ,$$

and hence by the previous paragraph to $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$. Thus $(\lambda, p) = 1$.

Proof of 2.1. Let N be an equivariant tubular neighborhood of B. Then

$$0 = H_*(\Sigma - K, N; \mathbf{Z}[1/\lambda]) = H_*(\Sigma - K - B, N - B; \mathbf{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$\begin{aligned} 0 = H_*((\Sigma - K - \mathrm{B})/G, (N - B)/G; \mathbf{Z}[1/\lambda]) &= H_*((\Sigma - K)/G, N/G; \mathbf{Z}[1/\lambda]) \\ &= H_*((\Sigma - K)/G, B; \mathbf{Z}[1/\lambda]) , \end{aligned}$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence $B \to N/G$. Thus $H_*(\bar{\Sigma} - \bar{K})$ looks like $H_*(S^1)$ except possibly for some λ-torsion. But by 2.1, λ is prime to the order of G, so for all primes q dividing λ, the transfer map $\mathrm{tr}: H_*(\bar{\Sigma} - \bar{K}; \mathbf{Z}/q) \to H_*(\Sigma - K; \mathbf{Z}/q)$ is injective so there is no extra λ-torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let X and $\bar{X}$ be the infinite cyclic covers of $\Sigma - K$ and $\bar{\Sigma} - \bar{K}$ respectively. Let $\Delta_K(t) = \Pi_{i>0}[H_i(X)]^{(-1)^{i+1}}$ and $\Delta_{\bar{K}}(t) = \Pi_{i>0}[H_i(\bar{X})]^{(-1)^{i+1}}$. The Wang sequence shows that multiplication by $t-1$ induces an isomorphism on $H_i(X)$ for $i > 0$, so that if we take the polynomial represented by $[H_i(X)]$ and plug in $t = 1$ we get ± 1. (Indeed if we consider the ring homomorphism $\varphi : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}$ defined by $\varphi(t) = 1$, then $\varphi([H_i(X)])$ is a divisor of $[H_i(X) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$.) Thus $[H_i(X)]$ represented a non-zero element in $\mathbf{F}_p[t, t^{-1}]$, and hence $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ give well-defined elements of $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$. Then the considerations of §1 show:

THEOREM 2.3. *Let K be a G-periodic knot in a homology q-sphere Σ with fixed set B, where G is a group of prime power order p^r. Let λ be the linking number of K and B. Then*

$$\Delta_K(t) \doteqdot \Delta_{\bar{K}}(t)^{p^r}(1 + t + \dots + t^{\lambda - 1})^{p^{r-1}} \pmod p .$$

§3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any $\lambda \equiv \pm 1 \pmod 8$, T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number λ in a homology 3-sphere Ω whose $C_2 \times C_2$-cover branched over the link is a homology 3-sphere Σ. We will show that this congruence condition is necessary. Equivalently, we show

THEOREM 3.1. *Suppose the Klein 4-group $G \times H \cong C_2 \times C_2$ acts on a homology 3-sphere Σ so that the fixed sets Σ^G and Σ^H are disjoint circles. Then their linking number λ is congruent to ± 1 modulo 8.*

Proof. We have

$$\begin{array}{ccc} \Sigma & \to & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \to & \Sigma/(G \times H) . \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let $K = \Sigma^G/G \subset \Sigma/G$ and $\bar{K} = K/H \subset \Sigma/(G \times H)$. Then K is a knot of period 2. Renormalize $\Delta_K(t)$ and $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$ so that $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$, and $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$. Murasugi's congruence shows