Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PERIODIC KNOTS, SMITH THEORY, AND MURASUGI'S CONGRUENCE
Autor: Davis, James F. / Livingston, Charles
Kapitel: §2. HIGH-DIMENSIONAL PERIODIC KNOTS
DOI: https://doi.org/10.5169/seals-58725

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$$\sigma = 1 + g + g^2 + \ldots + g^{p-1}$$
$$\delta = 1 - g$$

be elements of the group ring $\mathbf{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbf{F}_p[t, t^{-1}]$-modules (all homology is with $\mathbf{F}_p$-coefficients).

$$\begin{aligned}
&0 \to C_*(X^G) \to C_*(\bar{X}) \xrightarrow{\mathrm{tr}} \sigma C_*(X) \to 0\\
&0 \to \delta C_*(X) \oplus C_*(X^G) \to C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \to 0\\
&0 \to \sigma C_*(X) \to \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \to 0\\
&\qquad\qquad \vdots\\
&0 \to \sigma C_*(X) \to \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \to 0\,.
\end{aligned}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbf{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbf{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X))$. The above homological considerations show

$$\begin{aligned}
\chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X)\\
\chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X)\\
\chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X)\\
&\vdots\\
\chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X)\,.
\end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p\,.$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1}\,.$$

Finally suppose G has order p^r. Let G_1 be a normal subgroup of index p. By the exact sequences above $\mathrm{rk}\, H_*(X/G_1; \mathbf{F}_p) < \infty$. By applying inductively the result for the G_1-action on X and the G/G_1 action on X/G_1, Theorem B follows.

§2. High-dimensional periodic knots

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots

were introduced in the thesis of *R.* Cruz [C]. He showed that if there is a semifree $\mathbf{Z}/q$-action on S^n with non-empty fixed set and an invariant knot K^{n-2} disjoint from the fixed set, then the fixed set is S^1 if $q \neq 2$, and is S^1 or S^0 if $q = 2$.

For our purposes a knot K in a homology n-sphere Σ is an embedded $(n-2)$-dimensional homology sphere. Let G be a finite group. The knot K is *G-periodic* if it is invariant under a semifree G-action on Σ with fixed set $B \cong S^1$ disjoint from K. To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient $\bar{\Sigma} = \Sigma/G$ will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

PROPOSITION 2.1. $H_*(\bar{\Sigma} - \bar{K}) \cong H_*(S^1)$.

First we need a lemma.

LEMMA 2.2. *The linking number* $\lambda = \mathrm{lk}(B, K)$ *is relatively prime to the order of* G.

Proof. (See also [C, 2.1.1]). By restricting the action to a subgroup $\mathbf{Z}/p$ of G, we will assume $G = \mathbf{Z}/p$, and show $(\lambda, p) = 1$. By applying the Lefschetz Fixed-Point Theorem to a generator g of $\mathbf{Z}/p$, we see that if n is odd, the action on K is orientation-preserving, while if n is even, then $p = 2$ and the action is orientation-reversing. For local coefficients we will use $\mathbf{Z}^t$, the integers with the $\mathbf{Z}[\mathbf{Z}/p]$-module structure given by $(\Sigma a_i g^i) \cdot k = \Sigma a_i (-1)^{i(n+1)} k$.

Let $\bar{\Sigma} - B \to K(\mathbf{Z}/p, 1)$ classify the G-cover. We will consider the commutative diagram:

$$
\begin{array}{ccccc}
 & H_{n-2}(K;\mathbf{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K};\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t) \\
(*) & \downarrow & & \downarrow & & \| \\
 & H_{n-2}(\Sigma - B;\mathbf{Z}) & \to & H_{n-2}(\bar{\Sigma} - B;\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t)\,.
\end{array}
$$

The two groups on the left are infinite cyclic and the left vertical map is multiplication by λ. A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$.

The map α is isomorphic to $\mathbf{Z} \xrightarrow{\times p} \mathbf{Z}$ because it comes from a p-fold cover of $(n-2)$-dimensional closed manifolds. The map

$$H_{n-2}(\bar{K};\mathbf{Z}^t) \to H_{n-2}(\mathbf{Z}/p;\mathbf{Z}^t)$$

we compute algebraically by using a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$ as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on $\bar{K}$ to K,

$$C_*(K) = \{C_{n-2} \to \ldots \to C_0\}$$

with the i-chains C_i free $\mathbf{Z}G$-modules. By mapping a free $\mathbf{Z}G$-module onto $\ker(C_{n-2} \to C_{n-3})$ and continuing inductively, one constructs a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$

$$D_* = \{\ldots \to D_n \to D_{n-1} \to C_{n-2} \to \ldots \to C_0\} .$$

It follows that

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) = H_{n-2}(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}^t)$$

maps onto $H_{n-2}(D_* \otimes_{\mathbf{Z}G} \mathbf{Z}^t) = H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$. Furthermore by using the standard $\mathbf{Z}G$-resolution of $\mathbf{Z}$ (see e.g. [Mac]), one easily computes that $H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t) \cong \mathbf{Z}/p$.

Choose a G-invariant normal disk to B in Σ and let S^{n-2} be its boundary. Then the inclusion $S^{n-2} \to \Sigma - B$ is a homology equivalence. By the comparison theorem applied to the spectral sequence of the G-coverings (see [Mac]), the bottom row of (*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbf{Z}) \to H_{n-2}(S^{n-2}/G; \mathbf{Z}^t) \to H_{n-2}(G; \mathbf{Z}^t) ,$$

and hence by the previous paragraph to $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$. Thus $(\lambda, p) = 1$.

Proof of 2.1. Let N be an equivariant tubular neighborhood of B. Then

$$0 = H_*(\Sigma - K, N; \mathbf{Z}[1/\lambda]) = H_*(\Sigma - K - B, N - B; \mathbf{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$0 = H_*((\Sigma - K - \mathrm{B})/G, (N - B)/G; \mathbf{Z}[1/\lambda]) = H_*((\Sigma - K)/G, N/G; \mathbf{Z}[1/\lambda])$$
$$= H_*((\Sigma - K)/G, B; \mathbf{Z}[1/\lambda]) ,$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence $B \to N/G$. Thus $H_*(\bar{\Sigma} - \bar{K})$ looks like $H_*(S^1)$ except possibly for some λ-torsion. But by 2.1, λ is prime to the order of G, so for all primes q dividing λ, the transfer map $\mathrm{tr}: H_*(\bar{\Sigma} - \bar{K}; \mathbf{Z}/q) \to H_*(\Sigma - K; \mathbf{Z}/q)$ is injective so there is no extra λ-torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let X and $\bar{X}$ be the infinite cyclic

covers of $\Sigma - K$ and $\bar{\Sigma} - \bar{K}$ respectively. Let $\Delta_K(t) = \Pi_{i>0}[H_i(X)]^{(-1)^{i+1}}$ and $\Delta_{\bar{K}}(t) = \Pi_{i>0}[H_i(\bar{X})]^{(-1)^{i+1}}$. The Wang sequence shows that multiplication by $t-1$ induces an isomorphism on $H_i(X)$ for $i > 0$, so that if we take the polynomial represented by $[H_i(X)]$ and plug in $t = 1$ we get ± 1. (Indeed if we consider the ring homomorphism $\varphi: \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}$ defined by $\varphi(t) = 1$, then $\varphi([H_i(X)])$ is a divisor of $[H_i(X) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$.) Thus $[H_i(X)]$ represented a non-zero element in $\mathbf{F}_p[t, t^{-1}]$, and hence $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ give well-defined elements of $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$. Then the considerations of §1 show:

THEOREM 2.3. *Let K be a G-periodic knot in a homology q-sphere Σ with fixed set B, where G is a group of prime power order p^r. Let λ be the linking number of K and B. Then*

$$\Delta_K(t) \doteq \Delta_{\bar{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda-1})^{p^r-1} \pmod{p} .$$

§3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any $\lambda \equiv \pm 1 \pmod 8$, T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number λ in a homology 3-sphere Ω whose $C_2 \times C_2$-cover branched over the link is a homology 3-sphere Σ. We will show that this congruence condition is necessary. Equivalently, we show

THEOREM 3.1. *Suppose the Klein 4-group $G \times H \cong C_2 \times C_2$ acts on a homology 3-sphere Σ so that the fixed sets Σ^G and Σ^H are disjoint circles. Then their linking number λ is congruent to ± 1 modulo 8.*

Proof. We have

$$\begin{array}{ccc} \Sigma & \to & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \to & \Sigma/(G \times H) . \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let $K = \Sigma^G/G \subset \Sigma/G$ and $\bar{K} = K/H \subset \Sigma/(G \times H)$. Then K is a knot of period 2. Renormalize $\Delta_K(t)$ and $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$ so that $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$, and $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$. Murasugi's congruence shows