

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PERIODIC KNOTS, SMITH THEORY, AND MURASUGI'S CONGRUENCE
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Kapitel: §1. Murasugi's Congruence
DOI: <https://doi.org/10.5169/seals-58725>

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(See [Mi] or §1 for definitions). We will be most interested in the case $F = \mathbf{F}_p$, the finite field with p elements.

THEOREM B. *Let G be a p -group. Suppose $C_\infty \times G$ act on a finite-dimensional CW complex X with $\text{rk } H_*(X; \mathbf{F}_p) < \infty$, so that G acts semifreely and cellularly. Then*

$$\chi_m(X; \mathbf{F}_p) \chi_m(X^G; \mathbf{F}_p)^{|G|-1} = \chi_m(X/G; \mathbf{F}_p)^{|G|}.$$

Applying this to the case where X is the infinite cyclic cover of $\Sigma - K$ will immediately yield Murasugi's congruence. One advantage of our approach is that it generalizes to the case of high-dimensional periodic knots.

In §1 we prove Theorem B and derive Theorem A. In §2 we discuss the high-dimensional case and in §3 give the following application of Murasugi's congruence to links.

PROPOSITION C. *Let L be a two-component link in a homology 3-sphere. If the $\mathbf{Z}/2 \times \mathbf{Z}/2$ - cover branched over the link is also a homology 3-sphere, then the linking number of the two components is congruent to ± 1 modulo 8.*

§1. MURASUGI'S CONGRUENCE

We will derive Theorem A from Theorem B and then prove Theorem B, but we first give some homological preliminaries. If R is a commutative Noetherian UFD with quotient field K and M is a finitely generated torsion R -module then we define the *order* of M to be $[M] = E^0(M) \in R/R^*$. Here we take an exact sequence

$$R^k \xrightarrow{A} R^m \rightarrow M \rightarrow 0,$$

and we let $E^0(M)$ be a greatest common divisor of the determinants of the $m \times m$ -submatrices of A . If M is a torsion f.g. R -module then $[M] \neq 0$, and we consider the order $[M]$ as an element of K^*/R^* . If

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence of torsion f.g. R -modules, then J. Levine [L, lemma 5] shows $[M] = [M'] [M'']$. It follows for formal reasons that if $C_* = \{C_n \rightarrow \dots \rightarrow C_0\}$ is a chain complex of torsion f.g. R -modules then

$$\chi_m(C_*) := \prod [C_i]^{(-1)^i}$$

equals $\chi_m(H_*(C_*))$. In particular if C_* is exact, then $\chi_m(C_*) = 1$.

Next we turn to Alexander polynomials. By Alexander duality $H_1(\Sigma - K) \cong \mathbf{Z}$. Let $\pi: X \rightarrow \Sigma - K$ be the infinite cyclic cover of the knot complement. The infinite cyclic group $C_\infty = \langle t \rangle$ acts on X and $H_1(X; \mathbf{Z})$ is a f.g. torsion module over the group ring $\mathbf{Z}[C_\infty] = \mathbf{Z}[t, t^{-1}]$. The Alexander polynomial $\Delta_K(t)$ is its associated order. (Note that $\mathbf{Z}[t, t^{-1}]^*$ consists of $\pm t^i$ and the quotient field of $\mathbf{Z}[t, t^{-1}]$ is the field of rational functions $\mathbf{Q}(t)$.) As usual we normalize so that $\Delta_K(t)$ is a polynomial with integer coefficients and non-zero constant term.

If K has period p^r , let $\bar{\pi}: \bar{X} \rightarrow \bar{\Sigma} - \bar{K}$ be the infinite cyclic cover of the quotient knot. The $G = \mathbf{Z}/p^r$ -action on $\Sigma - K$ lifts to a G -action on X with quotient \bar{X} and fixed set $\bar{B} = \pi^{-1}(B)$. Indeed, let g be a generator of G . Then $g \circ \pi: X \rightarrow \Sigma - K$ induces the trivial map on H_1 and so lifts to $\tilde{g}: X \rightarrow X$. Since g has a non-empty, path-connected fixed-point set there is a unique lift \tilde{g} with fixed points and the fixed point set is \bar{B} . Since \tilde{g}^{p^r} is a lift of the identity which has fixed points, it itself is the identity and hence \tilde{g} is a map of period p^r . This gives an action of $C_\infty \times G$ on X . It further follows that $X/G \rightarrow \bar{\Sigma} - \bar{K}$ is an abelian cover inducing the trivial map on H_1 , so that we can identify this cover with $\bar{\pi}$ and X/G with \bar{X} .

The cover π is classified by a map $c: \Sigma - K \rightarrow S^1 = K(\mathbf{Z}, 1)$ inducing an isomorphism on H_1 . The inclusion map $B \rightarrow \Sigma - K$ induces multiplication by the linking number λ on H_1 . Thus by considering $c|_B$ which classifies $\pi: \tilde{B} \rightarrow B$, we see \tilde{B} is homeomorphic to λ disjoint copies of \mathbf{R} , cyclically permuted by the action of C_∞ .

Now $H_i(X)$ and $H_i(\bar{X})$ are zero for $i > 1$ and $H_0(X)$ and $H_0(\bar{X})$ are isomorphic to $\mathbf{F}_p \cong \mathbf{F}_p[t, t^{-1}]/(t-1)\mathbf{F}_p[t, t^{-1}]$, so $\chi_m(X) = (t-1)/\Delta_K(t)$ and $\chi_m(\bar{X}) = (t-1)/\Delta_{\bar{K}}(t)$. Since $X^G = \bar{B}$ consists of λ arcs cyclically permuted by $C_\infty = \langle t \rangle$, $\chi(X^G) = t^\lambda - 1$. Putting this together with Theorem B we see

$$[(t-1)/\Delta_K(t)] [t^\lambda - 1]^{p^r-1} = [(t-1)/\Delta_{\bar{K}}(t)]^{p^r}$$

or $\Delta_K(t) = \Delta_{\bar{K}}(t)^{p^r} (1 + t + \dots + t^{\lambda-1})^{p^r-1}$ with the equality taking place in $\mathbf{F}_p(t)/\mathbf{F}_p[t, t^{-1}]^*$. This gives Murasugi's congruence.

Proof of Theorem B. We prove the theorem by induction on the order of G . Let G be a group of prime order p with generator g . Let

$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$

$$\delta = 1 - g$$

be elements of the group ring $\mathbb{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbb{F}_p[t, t^{-1}]$ -modules (all homology is with \mathbb{F}_p -coefficients).

$$\begin{aligned} 0 &\rightarrow C_*(X^G) \rightarrow C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \delta C_*(X) \oplus C_*(X^G) \rightarrow C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \rightarrow 0 \\ &\vdots \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \rightarrow 0. \end{aligned}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbb{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbb{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X)))$. The above homological considerations show

$$\begin{aligned} \chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\ \chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\ \chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\ &\vdots \\ \chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X). \end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p.$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1}.$$

Finally suppose G has order p^r . Let G_1 be a normal subgroup of index p . By the exact sequences above $\text{rk } H_*(X/G_1; \mathbb{F}_p) < \infty$. By applying inductively the result for the G_1 -action on X and the G/G_1 action on X/G_1 , Theorem B follows.

§2. HIGH-DIMENSIONAL PERIODIC KNOTS

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots