

# §1. Murasugi's Congruence

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(See [Mi] or §1 for definitions). We will be most interested in the case  $F = \mathbf{F}_p$ , the finite field with  $p$  elements.

**THEOREM B.** *Let  $G$  be a  $p$ -group. Suppose  $C_\infty \times G$  act on a finite-dimensional CW complex  $X$  with  $\text{rk } H_*(X; \mathbf{F}_p) < \infty$ , so that  $G$  acts semifreely and cellularly. Then*

$$\chi_m(X; \mathbf{F}_p) \chi_m(X^G; \mathbf{F}_p)^{|G|-1} = \chi_m(X/G; \mathbf{F}_p)^{|G|}.$$

Applying this to the case where  $X$  is the infinite cyclic cover of  $\Sigma - K$  will immediately yield Murasugi's congruence. One advantage of our approach is that it generalizes to the case of high-dimensional periodic knots.

In §1 we prove Theorem B and derive Theorem A. In §2 we discuss the high-dimensional case and in §3 give the following application of Murasugi's congruence to links.

**PROPOSITION C.** *Let  $L$  be a two-component link in a homology 3-sphere. If the  $\mathbf{Z}/2 \times \mathbf{Z}/2$ -cover branched over the link is also a homology 3-sphere, then the linking number of the two components is congruent to  $\pm 1$  modulo 8.*

## §1. MURASUGI'S CONGRUENCE

We will derive Theorem A from Theorem B and then prove Theorem B, but we first give some homological preliminaries. If  $R$  is a commutative Noetherian UFD with quotient field  $K$  and  $M$  is a finitely generated torsion  $R$ -module then we define the *order* of  $M$  to be  $[M] = E^0(M) \in R/R^*$ . Here we take an exact sequence

$$R^k \xrightarrow{A} R^m \rightarrow M \rightarrow 0,$$

and we let  $E^0(M)$  be a greatest common divisor of the determinants of the  $m \times m$ -submatrices of  $A$ . If  $M$  is a torsion f.g.  $R$ -module then  $[M] \neq 0$ , and we consider the order  $[M]$  as an element of  $K^*/R^*$ . If

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence of torsion f.g.  $R$ -modules, then J. Levine [L, lemma 5] shows  $[M] = [M'] [M'']$ . It follows for formal reasons that if  $C_* = \{C_n \rightarrow \dots \rightarrow C_0\}$  is a chain complex of torsion f.g.  $R$ -modules then

$$\chi_m(C_*) := \prod [C_i]^{(-1)^i}$$

equals  $\chi_m(H_*(C_*))$ . In particular if  $C_*$  is exact, then  $\chi_m(C_*) = 1$ .

Next we turn to Alexander polynomials. By Alexander duality  $H_1(\Sigma - K) \cong \mathbf{Z}$ . Let  $\pi: X \rightarrow \Sigma - K$  be the infinite cyclic cover of the knot complement. The infinite cyclic group  $C_\infty = \langle t \rangle$  acts on  $X$  and  $H_1(X; \mathbf{Z})$  is a f.g. torsion module over the group ring  $\mathbf{Z}[C_\infty] = \mathbf{Z}[t, t^{-1}]$ . The Alexander polynomial  $\Delta_K(t)$  is its associated order. (Note that  $\mathbf{Z}[t, t^{-1}]^*$  consists of  $\pm t^i$  and the quotient field of  $\mathbf{Z}[t, t^{-1}]$  is the field of rational functions  $\mathbf{Q}(t)$ .) As usual we normalize so that  $\Delta_K(t)$  is a polynomial with integer coefficients and non-zero constant term.

If  $K$  has period  $p^r$ , let  $\bar{\pi}: \bar{X} \rightarrow \bar{\Sigma} - \bar{K}$  be the infinite cyclic cover of the quotient knot. The  $G = \mathbf{Z}/p^r$ -action on  $\Sigma - K$  lifts to a  $G$ -action on  $X$  with quotient  $\bar{X}$  and fixed set  $\bar{B} = \pi^{-1}(B)$ . Indeed, let  $g$  be a generator of  $G$ . Then  $g \circ \pi: X \rightarrow \Sigma - K$  induces the trivial map on  $H_1$  and so lifts to  $\tilde{g}: X \rightarrow X$ . Since  $g$  has a non-empty, path-connected fixed-point set there is a unique lift  $\tilde{g}$  with fixed points and the fixed point set is  $\bar{B}$ . Since  $\tilde{g}^{p^r}$  is a lift of the identity which has fixed points, it itself is the identity and hence  $\tilde{g}$  is a map of period  $p^r$ . This gives an action of  $C_\infty \times G$  on  $X$ . It further follows that  $X/G \rightarrow \bar{\Sigma} - \bar{K}$  is an abelian cover inducing the trivial map on  $H_1$ , so that we can identify this cover with  $\bar{\pi}$  and  $X/G$  with  $\bar{X}$ .

The cover  $\pi$  is classified by a map  $c: \Sigma - K \rightarrow S^1 = K(\mathbf{Z}, 1)$  inducing an isomorphism on  $H_1$ . The inclusion map  $B \rightarrow \Sigma - K$  induces multiplication by the linking number  $\lambda$  on  $H_1$ . Thus by considering  $c|_B$  which classifies  $\pi: \bar{B} \rightarrow B$ , we see  $\bar{B}$  is homeomorphic to  $\lambda$  disjoint copies of  $\mathbf{R}$ , cyclically permuted by the action of  $C_\infty$ .

Now  $H_i(X)$  and  $H_i(\bar{X})$  are zero for  $i > 1$  and  $H_0(X)$  and  $H_0(\bar{X})$  are isomorphic to  $\mathbf{F}_p \cong \mathbf{F}_p[t, t^{-1}]/(t-1)\mathbf{F}_p[t, t^{-1}]$ , so  $\chi_m(X) = (t-1)/\Delta_K(t)$  and  $\chi_m(\bar{X}) = (t-1)/\Delta_{\bar{K}}(t)$ . Since  $X^G = \bar{B}$  consists of  $\lambda$  arcs cyclically permuted by  $C_\infty = \langle t \rangle$ ,  $\chi(X^G) = t^\lambda - 1$ . Putting this together with Theorem B we see

$$[(t-1)/\Delta_K(t)] [t^\lambda - 1]^{p^r - 1} = [(t-1)/\Delta_{\bar{K}}(t)]^{p^r}$$

or  $\Delta_K(t) = \Delta_{\bar{K}}(t)^{p^r} (1 + t + \dots + t^{\lambda-1})^{p^r - 1}$  with the equality taking place in  $\mathbf{F}_p(t)/\mathbf{F}_p[t, t^{-1}]^*$ . This gives Murasugi's congruence.

*Proof of Theorem B.* We prove the theorem by induction on the order of  $G$ . Let  $G$  be a group of prime order  $p$  with generator  $g$ . Let

$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$

$$\delta = 1 - g$$

be elements of the group ring  $\mathbf{F}_p[G]$ . Note that  $\delta\sigma = 0 = \sigma\delta$  and  $\delta^{p-1} = \sigma$ . We consider the following chain complexes of  $\mathbf{F}_p[t, t^{-1}]$ -modules (all homology is with  $\mathbf{F}_p$ -coefficients).

$$\begin{aligned} 0 &\rightarrow C_*(X^G) \rightarrow C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \delta C_*(X) \oplus C_*(X^G) \rightarrow C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \rightarrow 0 \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \rightarrow 0 \\ &\quad \vdots \\ 0 &\rightarrow \sigma C_*(X) \rightarrow \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \rightarrow 0. \end{aligned}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID  $\mathbf{F}_p[t, t^{-1}]$ . We use shorthand notation – if  $\rho \in \mathbf{F}_p[G]$ , we write  $\chi^\rho(X)$  instead of  $\chi(H_*(\rho C_*(X)))$ . The above homological considerations show

$$\begin{aligned} \chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\ \chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\ \chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\ &\quad \vdots \\ \chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X). \end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p.$$

Using the first equation to substitute for  $\chi^\sigma(X)$  one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1}.$$

Finally suppose  $G$  has order  $p^r$ . Let  $G_1$  be a normal subgroup of index  $p$ . By the exact sequences above  $\text{rk } H_*(X/G_1; \mathbf{F}_p) < \infty$ . By applying inductively the result for the  $G_1$ -action on  $X$  and the  $G/G_1$  action on  $X/G_1$ , Theorem B follows.

## §2. HIGH-DIMENSIONAL PERIODIC KNOTS

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots