

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 37 (1991)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON A THEOREM OF SIKORAV  
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**DOI:** <https://doi.org/10.5169/seals-58732>

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ON A THEOREM OF SIKORAV

by Marco BRUNELLA

The aim of this note is to give a new proof of the following theorem of J. C. Sikorav:

THEOREM ([Sik]). *Let  $M$  be a closed manifold, let*

$$\phi_t: T^*M \rightarrow T^*M, t \in [0, 1],$$

*be a hamiltonian isotopy and let  $L \subset T^*M$  be an immersed lagrangian submanifold with a generating function  $S: M \times \mathbf{R}^k \rightarrow \mathbf{R}$  quadratic at infinity, then also  $\phi_1(L)$  has a generating function quadratic at infinity.*

Recall that a *generating function* of an immersed lagrangian submanifold  $L \subset T^*M$  is a function  $S: M \times \mathbf{R}^k \rightarrow \mathbf{R}, (q, \lambda) \mapsto S(q, \lambda)$ , such that:

a)  $\frac{\partial S}{\partial \lambda}: M \times \mathbf{R}^k \rightarrow \mathbf{R}^k$  is transverse to  $0 \in \mathbf{R}^k$ , so  $\left\{ \frac{\partial S}{\partial \lambda} = 0 \right\} \subset M \times \mathbf{R}^k$

is a submanifold;

b)  $L = \{ \xi \in T^*M \mid \exists \lambda \in \mathbf{R}^k: \frac{\partial S}{\partial \lambda}(\pi(\xi), \lambda) = 0, \xi = d(S(\cdot, \lambda))(\pi(\xi)) \}$

where  $\pi: T^*M \rightarrow M$  is the canonical projection.

The generating function  $S$  is said to be *quadratic at infinity* if for some  $R > 0$

$$S(q, \lambda) = Q(\lambda) \quad \forall (q, \lambda) \in M \times \mathbf{R}^k, \|\lambda\| > R$$

where  $Q$  is some non-degenerate quadratic form.

A *hamiltonian isotopy* of a symplectic manifold is a smooth curve of symplectic diffeomorphisms  $\{ \phi_t \}_{t \in [0, 1]}$  such that

$$\phi_0 = id \quad \text{and} \quad \dot{\phi}_t \stackrel{\text{def}}{=} \frac{d}{ds} \Big|_{s=0} \phi_{t+s} \circ \phi_t^{-1}$$

is a hamiltonian vector field  $\forall t \in [0, 1]$ .

The above theorem is important in the intersection theory of lagrangian submanifolds.

Our proof starts with the following remark, contained in [Gir] under the name of "Chekanov trick". Let  $i: M \rightarrow \mathbf{R}^N$  be any embedding and let  $\mathbf{R}^N$ ,  $M$  be equipped with riemannian metrics such that  $i$  is an isometric embedding. These metrics induce isomorphisms  $T^*M \simeq TM$  and  $T^*\mathbf{R}^N \simeq T\mathbf{R}^N$ , so the embedding  $Ti: TM \rightarrow T\mathbf{R}^N$  induces an embedding

$$j: T^*M \rightarrow T^*\mathbf{R}^N$$

which is a symplectic embedding. Remark that  $j(T^*M)$  is a subbundle of  $(T^*\mathbf{R}^N)|_{i(M)}$  and there is a canonical decomposition

$$(T^*\mathbf{R}^N)|_{i(M)} = j(T^*M) \oplus N_{i(M)}^*$$

where  $N_{i(M)}^*$  is the conormal bundle of  $i(M)$  in  $\mathbf{R}^N$ . This decomposition is the dual version of the decomposition  $(T\mathbf{R}^N)|_{i(M)} = (Ti)(TM) \oplus N_{i(M)}$ ,  $N_{i(M)}$  being the normal bundle of  $i(M)$  in  $\mathbf{R}^N$ .

We want to extend hamiltonian isotopies and lagrangian submanifolds from  $T^*M (\simeq j(T^*M))$  to  $T^*\mathbf{R}^N$ .

LEMMA 1. Let  $\phi_t: T^*M \rightarrow T^*M$ ,  $t \in [0, 1]$ , be a hamiltonian isotopy and let  $j: T^*M \rightarrow T^*\mathbf{R}^N$  as above, then there exists a hamiltonian isotopy  $\psi_t: T^*\mathbf{R}^N \rightarrow T^*\mathbf{R}^N$ ,  $t \in [0, 1]$ , such that  $\forall t \in [0, 1]$ :

$$1) j \circ \phi_t = \psi_t \circ j$$

$$2) \psi_t \text{ leaves invariant } (T^*\mathbf{R}^N)|_{i(M)}$$

moreover, if  $V \subset \mathbf{R}^N$  is any neighborhood of  $i(M)$  then we may choose every  $\psi_t$  with support contained in  $(T^*\mathbf{R}^N)|_V$ .

*Proof.* Let  $pr: (T^*\mathbf{R}^N)|_{i(M)} = j(T^*M) \oplus N_{i(M)}^* \rightarrow j(T^*M)$  be the projection on the first factor;  $(T^*\mathbf{R}^N)|_{i(M)}$  is a coisotropic submanifold of  $T^*\mathbf{R}^N$  and the fibres of  $pr$  are its characteristic leaves. For every  $x \in j(T^*M)$  let  $E_x$  be the antiorthogonal complement of  $T_x(j(T^*M))$ ;  $E_x$  is transverse to  $T_x(j(T^*M))$  and intersects  $T_x((T^*\mathbf{R}^N)|_{i(M)})$  along  $T_x(pr^{-1}(x))$ . This implies that we may find a tubular neighborhood of  $j(T^*M)$  in  $T^*\mathbf{R}^N$ ,  $p_0: U \rightarrow j(T^*M)$ , such that the fibre  $p_0^{-1}(x)$  intersects  $(T^*\mathbf{R}^N)|_{i(M)}$  along  $pr^{-1}(x)$ :

$$p_0^{-1}(x) \cap (T^*\mathbf{R}^N)|_{i(M)} = pr^{-1}(x) \cap U \quad \forall x \in j(T^*M).$$

Let now  $\{H_t\}_{t \in [0, 1]}$  be hamiltonians of the vector fields  $\{\dot{\phi}_t\}_{t \in [0, 1]}$ , define  $\forall t \in [0, 1] K_t: U \cup (T^*\mathbf{R}^N)|_{i(M)} \rightarrow \mathbf{R}$  by:

$$K_t(x) = \begin{cases} H_t(j^{-1}(p_0(x))) & \text{if } x \in U \\ H_t(j^{-1}(pr(x))) & \text{if } x \in (T^*\mathbf{R}^N)|_{i(M)}. \end{cases}$$

The relation between  $p_0$  and  $pr$  guarantees that  $K_t$  are well defined; now extend smoothly the family  $\{K_t\}_{t \in [0,1]}$  on all  $T^*\mathbf{R}^N$ , in such a way that  $K_t$  are constant outside  $(T^*\mathbf{R}^N)|_V$  (this is possible choosing  $U$  such that its projection on  $\mathbf{R}^N$  has closure contained in  $V$ ). Then the hamiltonian isotopy  $\psi_t: T^*\mathbf{R}^N \rightarrow T^*\mathbf{R}^N$ ,  $t \in [0,1]$ , generated by  $\{K_t\}_{t \in [0,1]}$  satisfies the conclusions of the lemma.  $\square$

LEMMA 2. *Let  $L \subset T^*M$  be an immersed lagrangian submanifold with generating function  $S: M \times \mathbf{R}^k \rightarrow \mathbf{R}$  quadratic at infinity, then there exists an immersed lagrangian submanifold  $\tilde{L} \subset T^*\mathbf{R}^N$  with a generating function  $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \rightarrow \mathbf{R}$  quadratic at infinity and such that  $\tilde{L} \cap (T^*\mathbf{R}^N)|_{i(M)} = j(L)$  (transversally); moreover, if  $V \subset \mathbf{R}^N$  is any neighborhood of  $i(M)$  then we may choose  $\tilde{L}$  and  $\tilde{S}$  such that  $\tilde{L}$  is equal to the null section outside  $(T^*\mathbf{R}^N)|_V$  and  $\tilde{S}$  is equal to  $Q$  (= quadratic form associated to  $S$ ) outside  $V \times \mathbf{R}^k$ .*

*Proof.* Let  $W \xrightarrow{q_0} i(M)$  be a tubular neighborhood of  $i(M)$  in  $\mathbf{R}^N$ , with fibres orthogonal to  $i(M)$  and  $\bar{W} \subset V$ ; define

$$\tilde{S}: W \times \mathbf{R}^k \rightarrow \mathbf{R} \quad \text{by} \quad \tilde{S}(x, \lambda) = S(i^{-1}(q_0(x)), \lambda)$$

and extend  $\tilde{S}$  to all  $\mathbf{R}^N \times \mathbf{R}^k$  preserving the quadraticity at infinity and in such a way that  $\tilde{S}(x, \lambda) = Q(\lambda)$  outside  $V \times \mathbf{R}^k$ ; a transversality argument allows to find such a  $\tilde{S}$  such that it generates a lagrangian submanifold  $\tilde{L} \subset T^*\mathbf{R}^N$ , and clearly  $\tilde{L}, \tilde{S}$  satisfy the conclusions of the lemma.  $\square$

Conversely:

LEMMA 3. *Let  $\tilde{L} \subset T^*\mathbf{R}^N, L \subset T^*M$  be immersed lagrangian submanifolds such that  $j(L) = \tilde{L} \cap (T^*\mathbf{R}^N)|_{i(M)}$  transversally; if  $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \rightarrow \mathbf{R}$  is a generating function for  $\tilde{L}$ , then  $S: M \times \mathbf{R}^k \rightarrow \mathbf{R}, S(x, \lambda) = \tilde{S}(i(x), \lambda)$ , is a generating function for  $L$ .  $\square$*

Remark that if  $\tilde{S}$  is quadratic at infinity, so is  $S$ .

Let now  $L \subset T^*M, S: M \times \mathbf{R}^k \rightarrow \mathbf{R}, \phi_t: T^*M \rightarrow T^*M, t \in [0,1]$ , as in the hypotheses of the theorem. Let  $\psi_t: T^*\mathbf{R}^N \rightarrow T^*\mathbf{R}^N, t \in [0,1]$ , be an extension of  $\phi_t$  as in lemma 1, and let  $\tilde{L} \subset T^*\mathbf{R}^N$  be an extension of  $L$  as in lemma 2, with generating function  $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \rightarrow \mathbf{R}$ . We have:

$$j(\phi_1(L)) = \psi_1(j(L)) = \psi_1(\tilde{L}) \cap (T^*\mathbf{R}^N)|_{i(M)}$$

and so, by lemma 3, we have only to prove that  $\psi_1(\tilde{L})$  has a generating function, quadratic at infinity. If we choose  $V$  with compact closure, then the support properties of  $\psi_t$  outside  $(T^*\mathbf{R}^N)|_V$  allows to “compactify”  $\{\psi_t\}_{t \in [0,1]}$ , that is there exists a hamiltonian isotopy  $\{\tilde{\psi}_t\}_{t \in [0,1]}$  of  $T^*\mathbf{R}^N$  with compact support such that  $\tilde{\psi}_1(\tilde{L}) = \psi_1(\tilde{L})$  (remark that the quadraticity at infinity of  $\tilde{S}$  implies that  $\tilde{L} \cap (T^*\mathbf{R}^N)|_V$  is contained in a compact set, and so is  $[\cup_{t \in [0,1]} \psi_t(\tilde{L})] \cap (T^*\mathbf{R}^N)|_V$ ). Observe that  $\tilde{S}$  is equal to a quadratic form  $Q = Q(\lambda)$  outside a compact set.

We may decompose  $\tilde{\psi}_1$  as a product  $\tilde{\psi}_1 = g_1 \circ \dots \circ g_l$ , where each  $g_j$  is a symplectic diffeomorphism with compact support and with a *generating function*  $F_j: \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}$ :

$$(P, Q) = g_j(p, q) \Leftrightarrow P = p + \frac{\partial F_j}{\partial Q}(Q, p), q = Q + \frac{\partial F_j}{\partial p}(Q, p)$$

(we use here the standard symplectic coordinates on  $T^*\mathbf{R}^N$ ). We may suppose that each  $F_j$  has compact support.

The proof of the theorem is achieved by an iteration of the following lemma:

**LEMMA 4.** *Let  $\tilde{L} \subset T^*\mathbf{R}^N$  be an immersed lagrangian submanifold with generating function  $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \rightarrow \mathbf{R}$  such that outside a compact set  $\tilde{S}(x, \lambda) = Q(\lambda) =$  non-degenerate quadratic form, let  $g: T^*\mathbf{R}^N \rightarrow T^*\mathbf{R}^N$  be a symplectic diffeomorphism with a generating function  $F: \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}$  with compact support; then  $g(\tilde{L})$  has a generating function equal to a non-degenerate quadratic form outside a compact set.*

*Proof.* A computation shows that  $T: \mathbf{R}^N \times \mathbf{R}^k \times \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}$  defined by

$$T(x, \lambda, \xi, \eta) = F(x, \xi) + \tilde{S}(x + \eta, \lambda) - \xi \cdot \eta$$

is a generating function for  $g(\tilde{L})$  (with  $\lambda, \xi, \eta$  as parameters). Let  $\rho: [0, +\infty) \rightarrow [0, 1]$  be a smooth function s.t.  $\rho(t) = 1 \forall t \leq 1$  and  $\rho(t) = 0 \forall t \geq 2$ , then for an appropriate choice of positive constants  $K_1, K_2, K_3, K_4$  the function

$$\begin{aligned} \hat{T}(x, \lambda, \xi, \eta) &= \rho\left(\frac{\|\lambda\|}{K_1}\right) \rho\left(\frac{\|\eta\|}{K_2}\right) F(x, \xi) \\ &+ \rho\left(\frac{\|\xi\|}{K_3}\right) \rho\left(\frac{\|x\|}{K_4}\right) [\tilde{S}(x + \eta, \lambda) - Q(\lambda)] + Q(\lambda) - \xi \cdot \eta \end{aligned}$$

is again a generating function for  $g(\tilde{L})$ , and it is equal to the non degenerate quadratic form  $Q(\lambda) - \xi \cdot \eta$  outside a compact set. A possible choice of the constants  $K_j$  is the following:

$$\text{if } S_0(x, \lambda) \stackrel{\text{def}}{=} \tilde{S}(x, \lambda) - Q(\lambda),$$

$$\text{if } \text{supp } S_0 \subset \{\|x\| < R, \|\lambda\| < R\}, \text{supp } F \subset \{\|x\| < R, \|\xi\| < R\},$$

and

$$\text{if } a = \sup F, \quad b = \sup S_0, \quad c = \sup \left\| \frac{\partial F}{\partial \xi} \right\|, \quad d = \sup \left\| \frac{\partial S_0}{\partial x} \right\|,$$

$$e = \sup \left\| \frac{\partial S_0}{\partial \lambda} \right\|, \quad \beta = \sup \left| \frac{dp}{dt} \right|,$$

then define:

$$K_1 \text{ s.t. } K_1 > R \quad \text{and} \quad \|\lambda\| \geq K_1 \Rightarrow \left\| \frac{\partial Q}{\partial \lambda}(\lambda) \right\| > \frac{1}{K_1} \beta a + e$$

$$K_2 = K_3 = K \text{ s.t. } K > R, \quad K > \frac{1}{K} \beta b + c, \quad K > \frac{1}{K} \beta a + d$$

$$K_4 \text{ s.t. } K_4 > K + R. \quad \square$$

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(Reçu le 27 novembre 1990)

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