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ON A THEOREM OF SIKORAV

by Marco BRUNELLA

The aim of this note is to give a new proof of the following theorem of J.C. Sikorav:

THEOREM ([Sik]). *Let M be a closed manifold, let*

$$\phi_t : T^*M \to T^*M, \; t \in [0,1] ,$$

be a hamiltonian isotopy and let $L \subset T^*M$ be an immersed lagrangian submanifold with a generating function $S: M \times \mathbf{R}^k \to \mathbf{R}$ quadratic at infinity, then also $\phi_1(L)$ has a generating function quadratic at infinity.

Recall that a *generating function* of an immersed lagrangian submanifold $L \subset T^*M$ is a function $S: M \times \mathbf{R}^k \to \mathbf{R}$, $(q, \lambda) \mapsto S(q, \lambda)$, such that:

a) $\dfrac{\partial S}{\partial \lambda} : M \times \mathbf{R}^k \to \mathbf{R}^k$ is transverse to $0 \in \mathbf{R}^k$, so $\left\{ \dfrac{\partial S}{\partial \lambda} = 0 \right\} \subset M \times \mathbf{R}^k$ is a submanifold;

b) $L = \{ \xi \in T^*M \mid \exists \lambda \in \mathbf{R}^k : \dfrac{\partial S}{\partial \lambda}(\pi(\xi), \lambda) = 0, \xi = d(S(\cdot, \lambda))(\pi(\xi)) \}$

where $\pi : T^*M \to M$ is the canonical projection.

The generating function S is said to be *quadratic at infinity* if for some $R > 0$

$$S(q, \lambda) = Q(\lambda) \qquad \forall (q, \lambda) \in M \times \mathbf{R}^k, \; \| \lambda \| > R$$

where Q is some non-degenerate quadratic form.

A *hamiltonian isotopy* of a symplectic manifold is a smooth curve of symplectic diffeomorphisms $\{\phi_t\}_{t \in [0,1]}$ such that

$$\phi_0 = id \qquad \text{and} \qquad \dot{\phi}_t \overset{\text{def}}{=} \frac{d}{ds}\bigg|_{s=0} \phi_{t+s} \circ \phi_t^{-1}$$

is a hamiltonian vector field $\forall t \in [0,1]$.

The above theorem is important in the intersection theory of lagrangian submanifolds.

Our proof starts with the following remark, contained in [Gir] under the name of "Chekanov trick". Let $i: M \to \mathbf{R}^N$ be any embedding and let $\mathbf{R}^N$, M be equipped with riemannian metrics such that i is an isometric embedding. These metrics induce isomorphisms $T^*M \simeq TM$ and $T^*\mathbf{R}^N \simeq T\mathbf{R}^N$, so the embedding $Ti: TM \to T\mathbf{R}^N$ induces an embedding

$$j: T^*M \to T^*\mathbf{R}^N$$

which is a symplectic embedding. Remark that $j(T^*M)$ is a subbundle of $(T^*\mathbf{R}^N)|_{i(M)}$ and there is a canonical decomposition

$$(T^*\mathbf{R}^N)|_{i(M)} = j(T^*M) \oplus N^*_{i(M)}$$

where $N^*_{i(M)}$ is the conormal bundle of $i(M)$ in $\mathbf{R}^N$. This decomposition is the dual version of the decomposition $(T\mathbf{R}^N)|_{i(M)} = (Ti)(TM) \oplus N_{i(M)}$, $N_{i(M)}$ being the normal bundle of $i(M)$ in $\mathbf{R}^N$.

We want to extend hamiltonian isotopies and lagrangian submanifolds from $T^*M(\simeq j(T^*M))$ to $T^*\mathbf{R}^N$.

LEMMA 1. *Let* $\phi_t: T^*M \to T^*M, t \in [0,1]$, *be a hamiltonian isotopy and let* $j: T^*M \to T^*\mathbf{R}^N$ *as above, then there exists a hamiltonian isotopy* $\psi_t: T^*\mathbf{R}^N \to T^*\mathbf{R}^N$, $t \in [0,1]$, *such that* $\forall t \in [0,1]$:

1) $j \circ \phi_t = \psi_t \circ j$

2) ψ_t *leaves invariant* $(T^*\mathbf{R}^N)|_{i(M)}$

moreover, if $V \subset \mathbf{R}^N$ *is any neighborhood of* $i(M)$ *then we may choose every* ψ_t *with support contained in* $(T^*\mathbf{R}^N)|_V$.

Proof. Let $pr: (T^*\mathbf{R}^N)_{i(M)} = j(T^*M) \oplus N^*_{i(M)} \to j(T^*M)$ be the projection on the first factor; $(T^*\mathbf{R}^N)_{i(M)}$ is a coisotropic submanifold of $T^*\mathbf{R}^N$ and the fibres of pr are its characteristic leaves. For every $x \in j(T^*M)$ let E_x be the antiorthogonal complement of $T_x(j(T^*M))$; E_x is transverse to $T_x(j(T^*M))$ and intersects $T_x((T^*\mathbf{R}^N)|_{i(M)})$ along $T_x(pr^{-1}(x))$. This implies that we may find a tubular neighborhood of $j(T^*M)$ in $T^*\mathbf{R}^N$, $p_0: U \to j(T^*M)$, such that the fibre $p_0^{-1}(x)$ intersects $(T^*\mathbf{R}^N)|_{i(M)}$ along $pr^{-1}(x)$:

$$p_0^{-1}(x) \cap (T^*\mathbf{R}^N)|_{i(M)} = pr^{-1}(x) \cap U \quad \forall x \in j(T^*M) .$$

Let now $\{H_t\}_{t \in [0,1]}$ be hamiltonians of the vector fields $\{\dot{\phi}_t\}_{t \in [0,1]}$, define $\forall t \in [0,1]\, K_t: U \cup (T^*\mathbf{R}^N)|_{i(M)} \to \mathbf{R}$ by:

$$K_t(x) = \begin{cases} H_t(j^{-1}(p_0(x))) & \text{if} \quad x \in U \\ H_t(j^{-1}(pr(x))) & \text{if} \quad x \in (T^*\mathbf{R}^N)|_{i(M)} \, . \end{cases}$$

The relation between p_0 and pr guarantees that K_t are well defined; now extend smoothly the family $\{K_t\}_{t\in[0,1]}$ on all $T^*\mathbf{R}^N$, in such a way that K_t are constant outside $(T^*\mathbf{R}^N)|_V$ (this is possible choosing U such that its projection on $\mathbf{R}^N$ has closure contained in V). Then the hamiltonian isotopy $\psi_t: T^*\mathbf{R}^N \to T^*\mathbf{R}^N$, $t \in [0, 1]$, generated by $\{K_t\}_{t\in[0,1]}$ satisfies the conclusions of the lemma. □

LEMMA 2. *Let $L \subset T^*M$ be an immersed lagrangian submanifold with generating function $S: M \times \mathbf{R}^k \to \mathbf{R}$ quadratic at infinity, then there exists an immersed lagrangian submanifold $\tilde{L} \subset T^*\mathbf{R}^N$ with a generating function $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}$ quadratic at infinity and such that $\tilde{L} \cap (T^*\mathbf{R}^N)|_{i(M)} = j(L)$ (transversally); moreover, if $V \subset \mathbf{R}^N$ is any neighborhood of $i(M)$ then we may choose $\tilde{L}$ and $\tilde{S}$ such that $\tilde{L}$ is equal to the null section outside $(T^*\mathbf{R}^N)|_V$ and $\tilde{S}$ is equal to Q (= quadratic form associated to S) outside $V \times \mathbf{R}^k$.*

Proof. Let $W \overset{q_0}{\to} i(M)$ be a tubular neighborhood of $i(M)$ in $\mathbf{R}^N$, with fibres orthogonal to $i(M)$ and $\bar{W} \subset V$; define

$$\tilde{S}: W \times \mathbf{R}^k \to \mathbf{R} \quad \text{by} \quad \tilde{S}(x, \lambda) = S(i^{-1}(q_0(x)), \lambda)$$

and extend $\tilde{S}$ to all $\mathbf{R}^N \times \mathbf{R}^k$ preserving the quadraticity at infinity and in such a way that $\tilde{S}(x, \lambda) = Q(\lambda)$ outside $V \times \mathbf{R}^k$; a transversality argument allows to find such a $\tilde{S}$ such that it generates a lagrangian submanifold $\tilde{L} \subset T^*\mathbf{R}^N$, and clearly $\tilde{L}, \tilde{S}$ satisfy the conclusions of the lemma. □

Conversely:

LEMMA 3. *Let $\tilde{L} \subset T^*\mathbf{R}^N, L \subset T^*M$ be immersed lagrangian submanifolds such that $j(L) = \tilde{L} \cap (T^*\mathbf{R}^N)|_{i(M)}$ transversally; if $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}$ is a generating function for $\tilde{L}$, then $S: M \times \mathbf{R}^k \to \mathbf{R}, S(x, \lambda) = \tilde{S}(i(x), \lambda)$, is a generating function for L.* □

Remark that if $\tilde{S}$ is quadratic at infinity, so is S.

Let now $L \subset T^*M$, $S: M \times \mathbf{R}^k \to \mathbf{R}$, $\phi_t: T^*M \to T^*M$, $t \in [0, 1]$, as in the hypotheses of the theorem. Let $\psi_t: T^*\mathbf{R}^N \to T^*\mathbf{R}^N$, $t \in [0, 1]$, be an extension of ϕ_t as in lemma 1, and let $\tilde{L} \subset T^*\mathbf{R}^N$ be an extension of L as in lemma 2, with generating function $\tilde{S}: \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}$. We have:

$$j(\phi_1(L)) = \psi_1(j(L)) = \psi_1(\tilde{L}) \cap (T^*\mathbf{R}^N)|_{i(M)}$$

and so, by lemma 3, we have only to prove that $\psi_1(\tilde{L})$ has a generating function, quadratic at infinity. If we choose V with compact closure, then the support properties of ψ_t outside $(T^*\mathbf{R}^N)|_V$ allows to "compactify" $\{\psi_t\}_{t\in[0,1]}$, that is there exists a hamiltonian isotopy $\{\tilde{\psi}_t\}_{t\in[0,1]}$ of $T^*\mathbf{R}^N$ with compact support such that $\tilde{\psi}_1(\tilde{L}) = \psi_1(\tilde{L})$ (remark that the quadraticity at infinity of $\tilde{S}$ implies that $\tilde{L} \cap (T^*\mathbf{R}^N)|_V$ is contained in a compact set, and so is $[\cup_{t\in[0,1]}\psi_t(\tilde{L})] \cap (T^*\mathbf{R}^N)|_V$). Observe that $\tilde{S}$ is equal to a quadratic form $Q = Q(\lambda)$ outside a compact set.

We may decompose $\tilde{\psi}_1$ as a product $\tilde{\psi}_1 = g_1 \circ .. \circ g_l$, where each g_j is a symplectic diffeomorphism with compact support and with a *generating function* $F_j\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$:

$$(P, Q) = g_j(p, q) \Leftrightarrow P = p + \frac{\partial F_j}{\partial Q}(Q, p),\ q = Q + \frac{\partial F_j}{\partial p}(Q, p)$$

(we use here the standard symplectic coordinates on $T^*\mathbf{R}^N$). We may suppose that each F_j has compact support.

The proof of the theorem is achieved by an iteration of the following lemma:

LEMMA 4. *Let $\tilde{L} \subset T^*\mathbf{R}^N$ be an immersed lagrangian submanifold with generating function $\tilde{S}\colon \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}$ such that outside a compact set $\tilde{S}(x, \lambda) = Q(\lambda)$ = non-degenerate quadratic form, let $g\colon T^*\mathbf{R}^N \to T^*\mathbf{R}^N$ be a symplectic diffeomorphism with a generating function $F\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ with compact support; then $g(\tilde{L})$ has a generating function equal to a non-degenerate quadratic form outside a compact set.*

Proof. A computation shows that $T\colon \mathbf{R}^N \times \mathbf{R}^k \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ defined by

$$T(x, \lambda, \xi, \eta) = F(x, \xi) + \tilde{S}(x + \eta, \lambda) - \xi \cdot \eta$$

is a generating function for $g(\tilde{L})$ (with λ, ξ, η as parameters). Let $\rho\colon [0, +\infty) \to [0, 1]$ be a smooth function s.t. $\rho(t) = 1\ \forall t \leqslant 1$ and $\rho(t) = 0\ \forall t \geqslant 2$, then for an appropriate choice of positive constants K_1, K_2, K_3, K_4 the function

$$\hat{T}(x, \lambda, \xi, \eta) = \rho\left(\frac{\|\lambda\|}{K_1}\right)\rho\left(\frac{\|\eta\|}{K_2}\right)F(x, \xi)$$
$$+ \rho\left(\frac{\|\xi\|}{K_3}\right)\rho\left(\frac{\|x\|}{K_4}\right)[\tilde{S}(x + \eta, \lambda) - Q(\lambda)] + Q(\lambda) - \xi \cdot \eta$$

is again a generating function for $g(\tilde{L})$, and it is equal to the non degenerate quadratic form $Q(\lambda) - \xi \cdot \eta$ outside a compact set. A possible choice of the constants K_j is the following:

$$\text{if} \quad S_0(x, \lambda) \overset{\text{def}}{=} \tilde{S}(x, \lambda) - Q(\lambda) ,$$

$$\text{if} \quad \operatorname{supp} S_0 \subset \{\| x \| < R, \| \lambda \| < R\} , \ \operatorname{supp} F \subset \{\| x \| < R, \| \xi \| < R\} ,$$

and

$$\text{if} \quad a = \sup F, \quad b = \sup S_0, \quad c = \sup \left\| \frac{\partial F}{\partial \xi} \right\| , \quad d = \sup \left\| \frac{\partial S_0}{\partial x} \right\| ,$$

$$e = \sup \left\| \frac{\partial S_0}{\partial \lambda} \right\| , \quad \beta = \sup \left| \frac{d\rho}{dt} \right| ,$$

then define:

$$K_1 \text{ s.t. } K_1 > R \quad \text{and} \quad \| \lambda \| \geqslant K_1 \Rightarrow \left\| \frac{\partial Q}{\partial \lambda}(\lambda) \right\| > \frac{1}{K_1} \beta a + e$$

$$K_2 = K_3 = K \text{ s.t. } K > R, \ K > \frac{1}{K} \beta b + c, \ K > \frac{1}{K} \beta a + d$$

$$K_4 \text{ s.t. } K_4 > K + R . \quad \square$$

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