4. Any affinely regular minimal simplex has small faces

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3. List of small-faced affinely regular simplices

An integral simplex S with small faces is affinely regular if and only if the numerated simplices S_ν and $S_{\nu'}$ are equivalent for each pair ν, ν' of enumerations of S. In other terms, an integral simplex S with small faces of volume $k/n!$ is affinely regular if and only if $\rho_k(S_\nu) = \rho_k(S_{\nu'})$ for all enumerations ν and ν', hence if and only if the element $\rho_k(S_\nu)$ is a fixed point under the action of σ_{n+1} on $\rho_k(\Sigma_k)$.

It is sufficient for $\rho_k(S_\nu)$ to be fixed under the action of a set of generators in order to be a fixpoint of σ_{n+1} acting on $\rho_k(\Sigma_k)$. Let us suppose that $(\alpha_1, .., \alpha_{n-1})$ is a fixpoint of $\rho_k(\Sigma_k)$. Then, for all $i \in \{1, ..., n-2\}$:

$$\begin{aligned}(i, i+1) \cdot (\alpha_1, .., \alpha_i, \alpha_{i+1}, .., \alpha_{n-1}) &= (\alpha_1, .., \alpha_{i+1}, \alpha_i, .. \alpha_{n-1}) \\ &= (\alpha_1, .. \alpha_i, \alpha_{i+1}, .., \alpha_{n-1})\end{aligned}$$

implies $\alpha_i \equiv \alpha \pmod k$ for some $\alpha \in \mathbf{Z}/k\mathbf{Z}$.

Furthermore

$$(n-1, n) \cdot (\alpha, .., \alpha, \alpha) = (-\alpha\alpha^{-1}, .., -\alpha\alpha^{-1}, \alpha^{-1}) = (\alpha, .., \alpha)$$

gives $\alpha \equiv -\alpha\alpha^{-1} \equiv -1 \pmod k$.

Finally

$$(0, 1) \cdot (-1, -1, .., -1) = (1 - (n-1)(-1), -1, .., -1) = (-1, .., -1)$$

implies $-1 \equiv 1 - (n-1)(-1) \pmod k$ namely $0 \equiv n+1 \pmod k$ namely $k \mid (n+1)$.

This shows that the simplices listed in the theorem are exactly all the affinely regular simplices with small faces.

We have yet to show that any affinely regular minimal simplex is small-faced. This will be the aim of the next paragraph.

4. Any affinely regular minimal simplex has small faces

Lemma 1.2 implies the following corollary:

Corollary 4.1. *Every integral simplex of* $\mathbf{Z}^n$ *with numerated vertices is equivalent to an integral simplex with vertex* v_0 *at* 0 *and vertex* v_i *at the i-th vector-column of an upper triangular matrix* $(i > 0)$.

Regularity and Proposition 0.4 imply almost immediately the following:

Remark 4.2. Let S be an affinely regular simplex. Then S is minimal if and only if the interior of each edge of S is without integral points.

Let us start with the proof that each affinely regular minimal simplex is small-faced.

Consider an affinely regular minimal simplex S of $\mathbf{Z}^2$. Corollary 4.1 implies that S is equivalent to a simplex S' with vertices at 0 and at the vector-columns of a matrix of the type $\begin{pmatrix} l & a \\ 0 & k \end{pmatrix}$. Remark 4.2 implies that the integer l is equal to ± 1.

By exchanging S' with an equivalent simplex if necessary, we can suppose that S' has its vertices at 0 and at the vector-columns of a matrix of type $\begin{pmatrix} 1 & a \\ 0 & k \end{pmatrix}$ with k a positive integer. The affine regularity now implies that S' (and hence S) is small-faced.

Hence the theorem holds for $n = 2$.

Induction: $(n-1) \Rightarrow (n)$.

Let S_v be a numerated affinely regular minimal simplex of $\mathbf{Z}^n$ with underlying simplex S. Corollary 4.1 implies that, after some suitable choice of an equivalent simplex, we can suppose that $v_0 = 0$ and v_i is the i-th vector-column of an upper triangular $n \times n$ matrix T.

The $(n-1)$-face containing $v_0, v_1, .., v_{n-1}$ is an affinely regular simplex of $\mathbf{Z}^{n-1}$ and Remark 4.2 shows that it is minimal. So Lemma 1.2 and the induction hypothesis imply that, possibly after a suitable change of S_v, the matrix T is of the form

$$\begin{pmatrix} 1 & 0 & 0 & .. & 0 & l-1 & a_1 \\ 0 & 1 & 0 & .. & 0 & l-1 & a_2 \\ . & . & . & .. & . & . & . \\ 0 & 0 & 0 & .. & 1 & l-1 & a_{n-2} \\ 0 & 0 & 0 & .. & 0 & l & a_{n-1} \\ 0 & 0 & 0 & .. & 0 & 0 & k \end{pmatrix}$$

where l and k are positive integers, and where l divides n by induction hypothesis.

Set $\mu = n/l \in \mathbf{N}$.

The barycenter of $\mu v_0, \mu v_1, .., \mu v_{n-1}$ is $e_1 + e_2 + .. + e_{n-1}$.

Since S_v is affinely regular, there exists for all $i \in \{0, 1, .., n\}$ an element $g_i \in \mathrm{Aff}(\mathbf{Z}^n)$ which sends

$$\mu v_0, .., \mu v_{n-1}, \widehat{\mu v_n} \quad \text{to} \quad \mu v_0, .., \widehat{\mu v_i}, .., \mu v_n \ ;$$

the barycenter of $\mu v_0, .., \widehat{\mu v_i}, .., \mu v_n$, which is $g_i(e_1 + e_2 + .. + e_{n-1})$, is consequently also in $\mathbf{Z}^n$.

So the barycenters of all faces of μS_v are in $\mathbf{Z}^n$ and they are the vertices of an integral simplex S'.

Calculating the first coordinate of the barycenter of $\widehat{\mu v_0}, \mu v_1, .., \mu v_n$ we see that n divides $\mu + \mu(l-1) + \mu a_1$.

Calculating the first coordinate of the barycenter of $\mu v_0, \widehat{\mu v_1}, \mu v_2, .., \mu v_n$, we see that n divides $\mu(l-1) + \mu a_1$.

So the integer n divides μ too but this implies that $\mu = n$ and hence $l = 1$. This and the affine regularity imply that S is small-faced. □

The notions of affine regularity and of integrality may both be generalized to other polytopes, such as hypercubes, cross-polytopes, hexagones in dimension 2 or exceptional polytopes in dimension 4. We plan to consider these in a further paper.

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