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Autor: Bacher, Roland

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3. List of small-faced affinely regular simplices

An integral simplex S with small faces is affinely regular if and only if the numerated simplices S_ν and $S_{\nu'}$ are equivalent for each pair ν, ν' of enumerations of S. In other terms, an integral simplex S with small faces of volume $k/n!$ is affinely regular if and only if $\rho_k(S_\nu) = \rho_k(S_{\nu'})$ for all enumerations ν and ν', hence if and only if the element $\rho_k(S_\nu)$ is a fixed point under the action of σ_{n+1} on $\rho_k(\Sigma_k)$.

It is sufficient for $\rho_k(S_\nu)$ to be fixed under the action of a set of generators in order to be a fixpoint of σ_{n+1} acting on $\rho_k(\Sigma_k)$. Let us suppose that $(\alpha_1, .., \alpha_{n-1})$ is a fixpoint of $\rho_k(\Sigma_k)$. Then, for all $i \in \{1, ..., n-2\}$:

$$(i, i+1) \cdot (\alpha_1, .., \alpha_i, \alpha_{i+1}, .., \alpha_{n-1}) = (\alpha_1, .., \alpha_{i+1}, \alpha_i, .. \alpha_{n-1})$$
$$= (\alpha_1, .. \alpha_i, \alpha_{i+1}, .., \alpha_{n-1})$$

implies $\alpha_i \equiv \alpha \pmod{k}$ for some $\alpha \in \mathbf{Z}/k\mathbf{Z}$.

Furthermore

$$(n-1, n) \cdot (\alpha, .., \alpha, \alpha) = (-\alpha\alpha^{-1}, .., -\alpha\alpha^{-1}, \alpha^{-1}) = (\alpha, .., \alpha)$$

gives $\alpha \equiv -\alpha\alpha^{-1} \equiv -1 \pmod{k}$.

Finally

$$(0, 1) \cdot (-1, -1, .., -1) = (1 - (n-1)(-1), -1, .., -1) = (-1, .., -1)$$

implies $-1 \equiv 1 - (n-1)(-1) \pmod{k}$ namely $0 \equiv n + 1 \pmod{k}$ namely $k \mid (n+1)$.

This shows that the simplices listed in the theorem are exactly all the affinely regular simplices with small faces.

We have yet to show that any affinely regular minimal simplex is small-faced. This will be the aim of the next paragraph.

4. Any affinely regular minimal simplex has small faces

Lemma 1.2 implies the following corollary:

Corollary 4.1. *Every integral simplex of* $\mathbf{Z}^n$ *with numerated vertices is equivalent to an integral simplex with vertex* v_0 *at* 0 *and vertex* v_i *at the i-th vector-column of an upper triangular matrix* $(i > 0)$.

Regularity and Proposition 0.4 imply almost immediately the following: