

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AFFINELY REGULAR INTEGRAL SIMPLICES
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Kapitel: 1. Simplices with small faces
DOI: <https://doi.org/10.5169/seals-58731>

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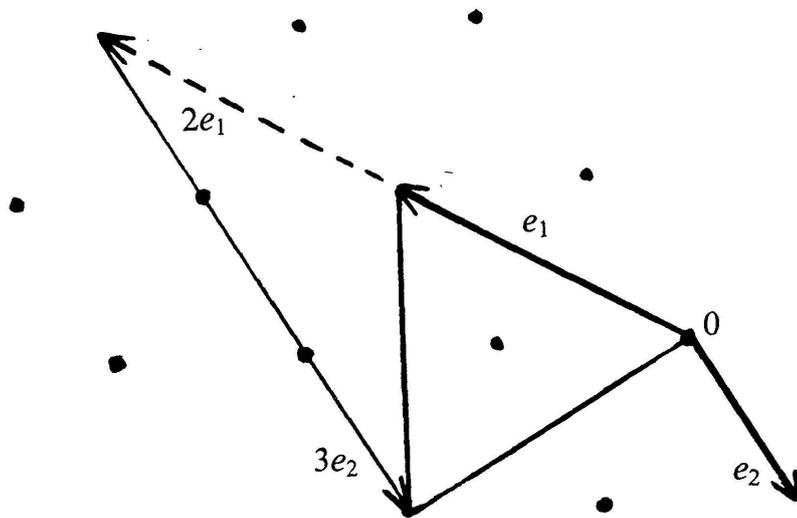
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In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0, 1]^3$ be the standard cube of \mathbf{R}^3 . Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3, \quad e_2 \mapsto e_1 + e_3, \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of \mathbf{R}^3) sends Δ to the representant given in Theorem 0.5.

1. SIMPLICES WITH SMALL FACES

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n - 1)$ -face of S , the vertices of S contained in H constitute an affine \mathbf{Z} -basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$v = (v_0, v_1, \dots, v_n)$$

of the vertices of S . We will denote by S_v the simplex S with numerotation v . The group $\text{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we

will say that S_v (with $v = (v_0, \dots, v_n)$) is equivalent to $S_{v'}$ (with $v' = (v'_0, \dots, v'_n)$) if there exists $g \in \text{Aff}(\mathbf{Z}^n)$ such that $g(v_i) = v'_i$ for all i .

The group σ_{n+1} acts on the set of numerated simplices: If $s \in \sigma_{n+1}$ is a permutation of $\{0, \dots, n\}$ and if S_v is an integral simplex numerated by $v = (v_0, \dots, v_i, \dots, v_n)$, we define

$$s \cdot S_v = S_{s \cdot v}$$

where

$$s \cdot v = (v_{s^{-1}(0)}, \dots, v_{s^{-1}(i)}, \dots, v_{s^{-1}(n)}) .$$

This action of σ_{n+1} commutes with that of $\text{Aff}(\mathbf{Z}^n)$, hence σ_{n+1} acts also on the equivalence classes of numerated simplices modulo $\text{Aff}(\mathbf{Z}^n)$.

An integral simplex S is affinely regular if and only if the stabilizer $\text{Stab}(S)$ operates transitively on the numerations of S .

Let us recall an elementary and well-known lemma:

LEMMA 1.2. *Let v_1, \dots, v_{n-1} be linearly independent vectors of \mathbf{Z}^n . Let H be the hyperplane of \mathbf{R}^n generated by the v_i 's, and suppose that v_1, \dots, v_{n-1} form a basis of the sublattice $H \cap \mathbf{Z}^n$.*

Then we can complete v_1, \dots, v_{n-1} to a basis of \mathbf{Z}^n .

Proof. See for instance the Corollary in Bourbaki, *Algèbre*, chap. VII, §4, No. 3.

From this point until the end of section 2, k will be some fixed natural integer. Let now S_v be a numerated simplex with small faces, of volume $k/n!$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbf{Z}^n \subset \mathbf{R}^n$. The lemma implies that there exists $g \in \text{Aff}(\mathbf{Z}^n)$ such that gS_v has vertices

$$v_0 = 0, v_1 = e_1, \dots, v_{n-1} = e_{n-1}, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i$$

where the a_i 's are integers. An easy calculation shows that the a_i 's are well defined (mod k).

Let us associate to S_v the element $(\alpha_1, \dots, \alpha_{n-1})$, where α_i is the class of a_i (mod k). This gives us a map $\rho_k: \Sigma_k \rightarrow (\mathbf{Z}/k\mathbf{Z})^{n-1}$, where Σ_k is the set of numerated simplices with small faces of volume $k/n!$. For $S_v \in \Sigma_k$, the element $\rho_k(S_v)$ depends only of the equivalence class of S_v modulo $\text{Aff}(\mathbf{Z}^n)$ and this allows us to define an action of σ_{n+1} on $\rho_k(\Sigma_k)$.