0. Introduction

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AFFINELY REGULAR INTEGRAL SIMPLICES

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0. INTRODUCTION

We will consider the standard lattice $\mathbf{Z}^n$ of the real vector space $\mathbf{R}^n$ with $n \geqslant 2$. An *integral simplex* is a non-degenerate simplex of $\mathbf{R}^n$ with all vertices in $\mathbf{Z}^n$. In this note, all simplices will be integral.

We will denote by $\mathrm{Aff}(\mathbf{Z}^n)$ the group of affine bijections of $\mathbf{R}^n$ which preserve $\mathbf{Z}^n$; it is the usual semi-direct product $\mathbf{Z}^n \rtimes GL_n(\mathbf{Z})$. The affine groupe $\mathrm{Aff}(\mathbf{Z}^n)$ acts naturally on the set of integral simplices in $\mathbf{Z}^n$.

For each integral simplex S we define

$$\mathrm{Stab}(S) = \{g \in \mathrm{Aff}(\mathbf{Z}^n) \mid g(S) = S\}$$

which is of course a subgroup of the group σ_{n+1} (group of permutations of $n+1$ objects), since there exists an injection in the group of permutations of the vertices of S.

Definition 0.1. A simplex S is called *affinely regular* if $\mathrm{Stab}(S)$ is equal to the whole group σ_{n+1}.

The definition of an affinely regular simplex is independent of the metric. For a discussion of integral simplices which are metrically regular one can consult [1] or [2] of the bibliography.

Two simplices S and S' are *equivalent* if there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that $g(S) = S'$. The scope of this note is to find all equivalence classes of affinely regular simplices.

Let S be a simplex. Let us denote by λS the image of the simplex S multiplied by some non-zero integer λ.

PROPOSITION 0.2. *The groups* $\mathrm{Stab}(S)$ *and* $\mathrm{Stab}(\lambda S)$ *are isomorphic for any integer* $\lambda \neq 0$.

Proof. Denote by $\delta(\lambda)$ the linear automorphism $x \mapsto \lambda x$ of $\mathbf{R}^n$. Let ϕ_λ denote the endomorphism $g \mapsto \delta(\lambda) g \delta(\lambda^{-1})$ of $\mathrm{Aff}(\mathbf{Z}^n)$; observe that ϕ_λ is one-to-one, but is not onto if $|\lambda| \geqslant 2$. Indeed, an affine bijection $g \in \mathrm{Aff}(\mathbf{Z}^n)$ is in the image of ϕ_λ if and only if g preserves the sublattice $\lambda \mathbf{Z}^n$ of $\mathbf{Z}^n$.

If $g \in \mathrm{Stab}(S)$, then $\phi_\lambda(g) \in \mathrm{Stab}(\lambda S)$. Consequently ϕ_λ restricts to an injective homomorphism $\psi_\lambda : \mathrm{Stab}(S) \to \mathrm{Stab}(\lambda S)$. Let now $h \in \mathrm{Stab}(\lambda S)$. We can write $h = at$, where a is in $GL_n(\mathbf{Z})$ and where t is a translation. As a^{-1} preserves $\lambda \mathbf{Z}^n$ (as any element of $GL_n(\mathbf{Z})$ does), and as h preserves λS one has

$$t(\lambda S) = a^{-1} h(\lambda S) = a^{-1}(\lambda S) \subset a^{-1} \lambda \mathbf{Z}^n$$

so that t preserves $\lambda \mathbf{Z}^n$. Hence $h = at$ preserves $\lambda \mathbf{Z}^n$, so that h is in the image of ϕ_λ. It follows that ψ_λ is an isomorphism onto. $\square$

Caution: We have in fact proved that $\mathrm{Stab}(S)$ and $\mathrm{Stab}(\lambda S)$ are conjugate in $\mathrm{Aff}(\mathbf{Q}^n)$ but they are in general not conjugate in $\mathrm{Aff}(\mathbf{Z}^n)$. This can be seen for instance by the fact that $\mathrm{Stab}(S)$ fixes the barycenter P of S and $\mathrm{Stab}(\lambda S)$ fixes λP. But P and λP are not necessarily in the same orbit of $\mathrm{Aff}(\mathbf{Z}^n)$.

So λS is affinely regular if and only if S is affinely regular. Hence we will be interested in minimal simplices.

Definition 0.3. An integral simplex S is *minimal* if, for every integral simplex T and for every integer $\lambda \geqslant 1$ such that S is equivalent to λT, we have $\lambda = 1$.

PROPOSITION 0.4. *Let S be an integral simplex of $\mathbf{Z}^n$. The following assertions are equivalent:*

i) *S is minimal.*

ii) *For every integer $\lambda \geqslant 2$ there exists no class of $\mathbf{Z}^n$ modulo $\lambda \mathbf{Z}^n$ which contains all the vertices of S modulo $\lambda \mathbf{Z}^n$.*

Proof. Not (ii) $\Rightarrow$ not (i). Let S be a simplex with all vertices in the same class of $\mathbf{Z}^n$ modulo $\lambda \mathbf{Z}^n$. Let v_0 be one of the vertices. The translate of S by $-v_0$ is then a simplex with the coordinates of all vertices divisible by some $\lambda \geqslant 2$. This implies that S is not minimal.

Not (i) $\Rightarrow$ not (ii). Let S be a non-minimal integral simplex. Hence there exists an integral simplex T, an integer $\lambda \geqslant 2$, an element $g \in GL_n(\mathbf{Z})$ and a vector $v \in \mathbf{Z}^n$ such that $S = g(\lambda T) + v$. But then all the vertices of S are in the class of v in $\mathbf{Z}^n$ modulo $\lambda \mathbf{Z}^n$. $\square$

The main subject of this note is to show the following theorem:

THEOREM 0.5. *For* $n \geqslant 2$, *one has a bijection between the equivalence classes of minimal affinely regular integral simplices and the set of positive divisors of* $n+1$ *(including* 1 *and* $n+1$*). The bijection associates to the divisor* k *of* $n+1$ *the class of the simplex whose vertices are given by the columns of the following* $n \times n$ *matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & k-1 \\ 0 & 1 & 0 & \dots & 0 & k-1 \\ . & . & . & \dots & . & k-1 \\ 0 & 0 & 0 & \dots & 1 & k-1 \\ 0 & 0 & 0 & \dots & 0 & k \end{pmatrix}$$

and by the origin of $\mathbf{Z}^n$.

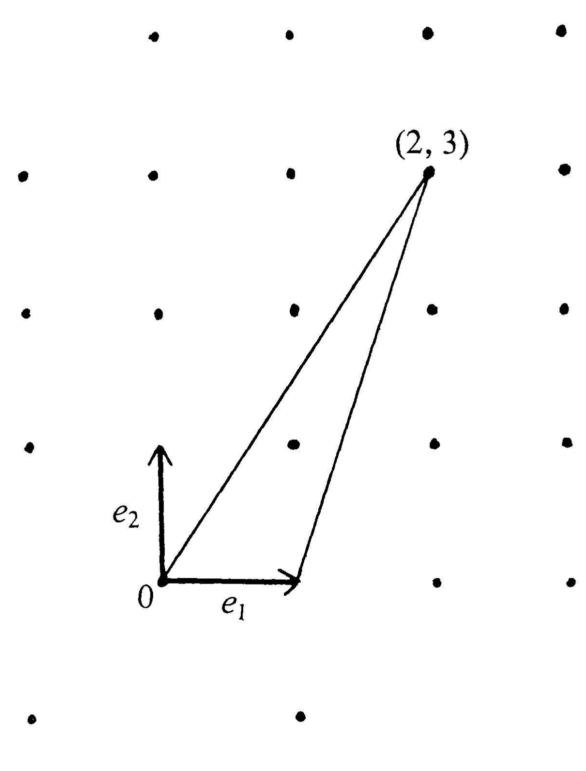
Proposition 0.4 implies that all representants in the theorem are minimal. Moreover, representants associated to distinct divisors k, k' of $n+1$ are non-equivalent since they are respectively of volumes $k/n!$ and $k'/n!$.

The plan of the proof is as follows. We will introduce a family of particular simplices: those which have small faces. Then we dress the list of all small-faced affinely regular simplices (this gives us in fact the list of the theorem). Last, we prove that an affinely regular minimal simplex is necessarily small-faced.

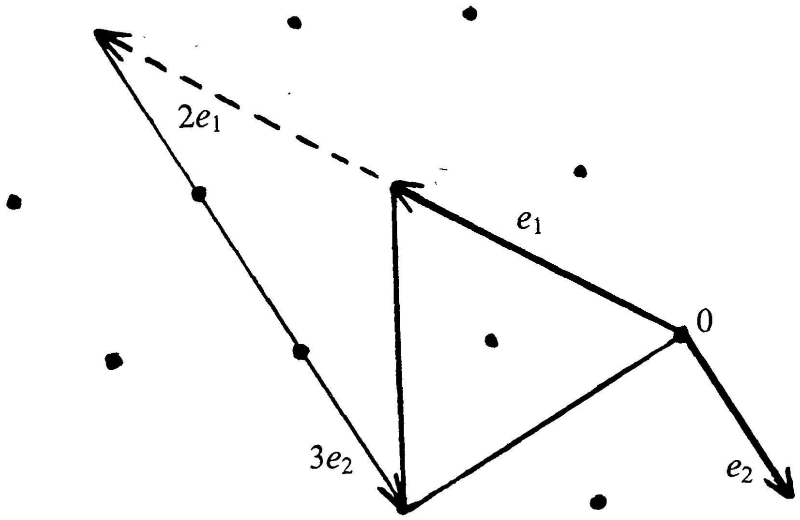
Let us start with some examples:

Example 0.6. Case where $n = 2, k = 3$.

In the standard lattice:



In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0, 1]^3$ be the standard cube of $\mathbf{R}^3$. Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3 , \quad e_2 \mapsto e_1 + e_3 , \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of $\mathbf{R}^3$) sends Δ to the representant given in Theorem 0.5.

1. Simplices with small faces

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n-1)$-face of S, the vertices of S contained in H constitute an affine $\mathbf{Z}$-basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$\nu = (v_0, v_1, .., v_n)$$

of the vertices of S. We will denote by S_ν the simplex S with numerotation ν. The group $\mathrm{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we