

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AFFINELY REGULAR INTEGRAL SIMPLICES
Autor: Bacher, Roland
Kapitel: 0. Introduction
DOI: <https://doi.org/10.5169/seals-58731>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

AFFINELY REGULAR INTEGRAL SIMPLICES

by Roland BACHER

0. INTRODUCTION

We will consider the standard lattice \mathbf{Z}^n of the real vector space \mathbf{R}^n with $n \geq 2$. An *integral simplex* is a non-degenerate simplex of \mathbf{R}^n with all vertices in \mathbf{Z}^n . In this note, all simplices will be integral.

We will denote by $\text{Aff}(\mathbf{Z}^n)$ the group of affine bijections of \mathbf{R}^n which preserve \mathbf{Z}^n ; it is the usual semi-direct product $\mathbf{Z}^n \rtimes GL_n(\mathbf{Z})$. The affine group $\text{Aff}(\mathbf{Z}^n)$ acts naturally on the set of integral simplices in \mathbf{Z}^n .

For each integral simplex S we define

$$\text{Stab}(S) = \{g \in \text{Aff}(\mathbf{Z}^n) \mid g(S) = S\}$$

which is of course a subgroup of the group σ_{n+1} (group of permutations of $n + 1$ objects), since there exists an injection in the group of permutations of the vertices of S .

Definition 0.1. A simplex S is called *affinely regular* if $\text{Stab}(S)$ is equal to the whole group σ_{n+1} .

The definition of an affinely regular simplex is independent of the metric. For a discussion of integral simplices which are metrically regular one can consult [1] or [2] of the bibliography.

Two simplices S and S' are *equivalent* if there exists $g \in \text{Aff}(\mathbf{Z}^n)$ such that $g(S) = S'$. The scope of this note is to find all equivalence classes of affinely regular simplices.

Let S be a simplex. Let us denote by λS the image of the simplex S multiplied by some non-zero integer λ .

PROPOSITION 0.2. *The groups $\text{Stab}(S)$ and $\text{Stab}(\lambda S)$ are isomorphic for any integer $\lambda \neq 0$.*

Proof. Denote by $\delta(\lambda)$ the linear automorphism $x \mapsto \lambda x$ of \mathbf{R}^n . Let ϕ_λ denote the endomorphism $g \mapsto \delta(\lambda)g\delta(\lambda^{-1})$ of $\text{Aff}(\mathbf{Z}^n)$; observe that ϕ_λ is

one-to-one, but is not onto if $|\lambda| \geq 2$. Indeed, an affine bijection $g \in \text{Aff}(\mathbf{Z}^n)$ is in the image of ϕ_λ if and only if g preserves the sublattice $\lambda\mathbf{Z}^n$ of \mathbf{Z}^n .

If $g \in \text{Stab}(S)$, then $\phi_\lambda(g) \in \text{Stab}(\lambda S)$. Consequently ϕ_λ restricts to an injective homomorphism $\psi_\lambda: \text{Stab}(S) \rightarrow \text{Stab}(\lambda S)$. Let now $h \in \text{Stab}(\lambda S)$. We can write $h = at$, where a is in $GL_n(\mathbf{Z})$ and where t is a translation. As a^{-1} preserves $\lambda\mathbf{Z}^n$ (as any element of $GL_n(\mathbf{Z})$ does), and as h preserves λS one has

$$t(\lambda S) = a^{-1}h(\lambda S) = a^{-1}(\lambda S) \subset a^{-1}\lambda\mathbf{Z}^n$$

so that t preserves $\lambda\mathbf{Z}^n$. Hence $h = at$ preserves $\lambda\mathbf{Z}^n$, so that h is in the image of ϕ_λ . It follows that ψ_λ is an isomorphism onto. \square

Caution: We have in fact proved that $\text{Stab}(S)$ and $\text{Stab}(\lambda S)$ are conjugate in $\text{Aff}(\mathbf{Q}^n)$ but they are in general not conjugate in $\text{Aff}(\mathbf{Z}^n)$. This can be seen for instance by the fact that $\text{Stab}(S)$ fixes the barycenter P of S and $\text{Stab}(\lambda S)$ fixes λP . But P and λP are not necessarily in the same orbit of $\text{Aff}(\mathbf{Z}^n)$.

So λS is affinely regular if and only if S is affinely regular. Hence we will be interested in minimal simplices.

Definition 0.3. An integral simplex S is *minimal* if, for every integral simplex T and for every integer $\lambda \geq 1$ such that S is equivalent to λT , we have $\lambda = 1$.

PROPOSITION 0.4. *Let S be an integral simplex of \mathbf{Z}^n . The following assertions are equivalent:*

- i) S is minimal.
- ii) For every integer $\lambda \geq 2$ there exists no class of \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$ which contains all the vertices of S modulo $\lambda\mathbf{Z}^n$.

Proof. Not (ii) \Rightarrow not (i). Let S be a simplex with all vertices in the same class of \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$. Let v_0 be one of the vertices. The translate of S by $-v_0$ is then a simplex with the coordinates of all vertices divisible by some $\lambda \geq 2$. This implies that S is not minimal.

Not (i) \Rightarrow not (ii). Let S be a non-minimal integral simplex. Hence there exists an integral simplex T , an integer $\lambda \geq 2$, an element $g \in GL_n(\mathbf{Z})$ and a vector $v \in \mathbf{Z}^n$ such that $S = g(\lambda T) + v$. But then all the vertices of S are in the class of v in \mathbf{Z}^n modulo $\lambda\mathbf{Z}^n$. \square

The main subject of this note is to show the following theorem:

THEOREM 0.5. For $n \geq 2$, one has a bijection between the equivalence classes of minimal affinely regular integral simplices and the set of positive divisors of $n + 1$ (including 1 and $n + 1$). The bijection associates to the divisor k of $n + 1$ the class of the simplex whose vertices are given by the columns of the following $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & k-1 \\ 0 & 1 & 0 & \dots & 0 & k-1 \\ \cdot & \cdot & \cdot & \dots & \cdot & k-1 \\ 0 & 0 & 0 & \dots & 1 & k-1 \\ 0 & 0 & 0 & \dots & 0 & k \end{pmatrix}$$

and by the origin of \mathbf{Z}^n .

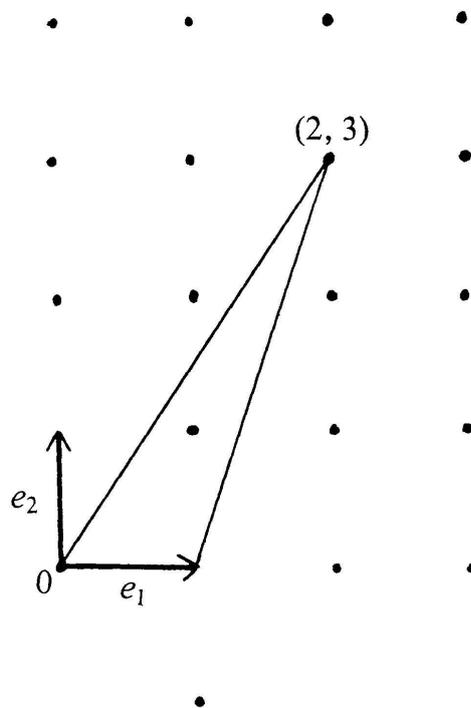
Proposition 0.4 implies that all representants in the theorem are minimal. Moreover, representants associated to distinct divisors k, k' of $n + 1$ are non-equivalent since they are respectively of volumes $k/n!$ and $k'/n!$.

The plan of the proof is as follows. We will introduce a family of particular simplices: those which have small faces. Then we dress the list of all small-faced affinely regular simplices (this gives us in fact the list of the theorem). Last, we prove that an affinely regular minimal simplex is necessarily small-faced.

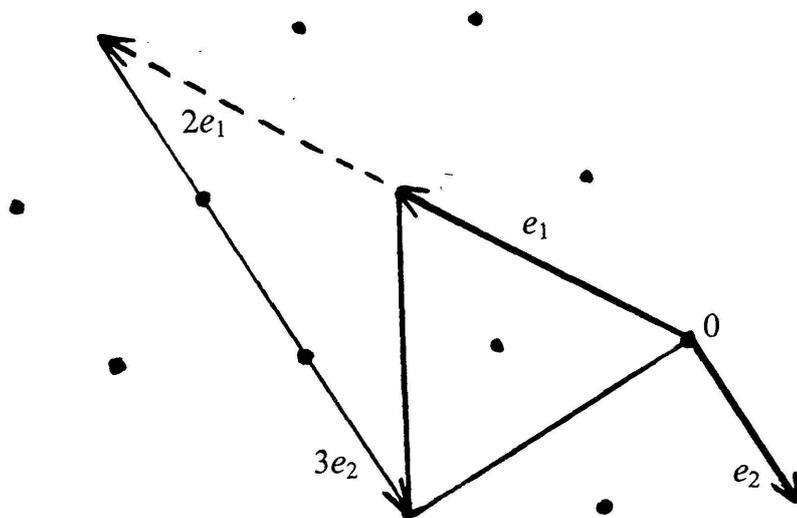
Let us start with some examples:

Example 0.6. Case where $n = 2, k = 3$.

In the standard lattice:



In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0, 1]^3$ be the standard cube of \mathbf{R}^3 . Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3, \quad e_2 \mapsto e_1 + e_3, \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of \mathbf{R}^3) sends Δ to the representant given in Theorem 0.5.

1. SIMPLICES WITH SMALL FACES

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n - 1)$ -face of S , the vertices of S contained in H constitute an affine \mathbf{Z} -basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$v = (v_0, v_1, \dots, v_n)$$

of the vertices of S . We will denote by S_v the simplex S with numerotation v . The group $\text{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we