3. The proof of Theorem B

It is clear that the subring $C(Q)$ of the integral cohomology $H^*(Q, \mathbf{Z})$ generated by (the classes of) $c_1, \ldots, c_n$ has the defining relations

$$l_i c_i = 0, \quad 1 \leqslant i \leqslant n. \tag{3.1.1}$$

Since the c_i are Chern classes of the obvious 1-dimensional complex representations of Q we refer to $C(Q)$ as the *Chern ring* of Q.

Theorem 3.1. *As a module over its Chern ring, the integral cohomology $H^*(Q, \mathbf{Z})$ of a finite abelian group $Q = C_{l_1} \times \cdots \times C_{l_n}$ with $l_1 \mid l_2 \mid \ldots \mid l_n$ is generated by 1 and the classes $\zeta_{x_i x_j \ldots x_k}$ of the kind (1.5) with $x_i x_j \ldots x_k$ of degree at least two, subject to the relations*

$$l_i \zeta_{x_i x_j \ldots x_k} = 0. \tag{3.1.2}$$

Proof. We prove the Theorem by induction. It is clear that when Q is cyclic there is no monomial $\zeta_{x_i x_j \ldots x_k}$ of the kind (1.5) with $x_i x_j \ldots x_k$ of degree at least two and hence there is nothing to prove. Thus the induction starts.

Next, let

$$G = C_{l_1} \times \cdots \times C_{l_n} = < t_1, \ldots, t_n; \quad t_j^{l_j} = 1 >, \quad \text{with} \quad l_1 \mid l_2 \mid \ldots \mid l_n,$$

let

$$Q = G \times \mathbf{Z}/l = < t_1, \ldots, t_n, t; t_j^{l_j} = 1, t^l = 1 >,$$

and suppose that the exponent of G divides l, that is,

$$l_1 \mid l_2 \mid \ldots \mid l_n \mid l.$$

It is manifest that the model $\mathscr{A}(Q)$ may be written

$$\mathscr{A}(Q) = \mathscr{A}(G) \otimes \mathscr{A}(\mathbf{Z}/l).$$

Regard the cycles $C(\mathscr{A}(\mathbf{Z}/l))$ and the boundaries $B(\mathscr{A}(\mathbf{Z}/l))$ as complexes with zero differential, and write $D(\mathscr{A}(\mathbf{Z}/l))$ for the boundaries $B(\mathscr{A}(\mathbf{Z}/l))$, regraded up by one, so that the exact sequence

$$0 \to C(\mathscr{A}(\mathbf{Z}/l)) \xrightarrow{\kappa} \mathscr{A}(\mathbf{Z}/l) \to D(\mathscr{A}(\mathbf{Z}/l)) \to 0$$

of chain complexes results. Since $\mathscr{A}(G)$ is free as a graded abelian group,

$$0 \to \mathscr{A}(G) \otimes C(\mathscr{A}(\mathbf{Z}/l)) \xrightarrow{\mathscr{A}(G) \otimes \kappa} \mathscr{A}(G) \otimes \mathscr{A}(\mathbf{Z}/l) \to \mathscr{A}(G) \otimes D(\mathscr{A}(\mathbf{Z}/l)) \to 0$$

is an exact sequence of chain complexes, too. In the standard way, cf. e.g. what is said on p. 166 of Mac Lane [23], its homology exact sequence boils down to the Künneth exact sequence

$$0 \to H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z}) \to H^*(Q, \mathbf{Z}) \to \mathrm{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z})) \to 0 .$$

It is well known that this sequence splits. Exploiting the inductive hypothesis we conclude at once that, as a module over the Chern ring $C(Q)$, $H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \dots x_k}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least two, subject to the relations

$$l_i \zeta_{x_i x_j \dots x_k} = 0 .$$

Likewise, as a module over the Chern ring $C(Q)$, $\mathrm{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ is generated by the images in $\mathrm{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ of the classes $\zeta_{x_i x_j \dots x_k x}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least one, subject to the relations

$$l_i \zeta_{x_i x_j \dots x_k} = 0 .$$

Since the Künneth sequence splits, this completes the proof. □

We note that the above generators $\zeta_{x_i x_j \dots x_k}$ can presumably be understood in terms of the multi torsion product given in Mac Lane [25] generalizing the triple torsion product introduced in Mac Lane [24]. Details have not been worked out yet.

We now refer to the subring of $A(Q)$ generated by $\tilde{\zeta}_{x_1}, \tilde{\zeta}_{x_2}, \dots, \tilde{\zeta}_{x_n}$ as the *Chern ring* of $A(Q)$. It is clear that (1.13) identifies the Chern rings.

Proof of Theorem B. In view of (3.1), as a module over the Chern ring $C(Q)$, $H^*(Q, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \dots x_k}$ with $x_i x_j \dots x_k$ of degree at least two; hence (1.13) is an isomorphism over the Chern ring. Furthermore, (1.13) is induced by the restriction of (1.11) to the cycles in $\mathscr{A}(Q)$. Since the product structure in the cohomology ring is induced by the product structure in $\mathscr{A}(Q)$, and since the algebra $A(Q)$ arises from $\mathscr{A}(Q)$ by introducing the additional relations $l_i c_i = 0$, the association (1.13) identifies $H^*(Q, \mathbf{Z})$ with the subalgebra of $A(Q)$ generated by the $\tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$. □

Under the circumstances of Theorem B it is straightforward to work out explicit formulas for the products

$$\tilde{\zeta}_{x_i x_j \dots x_k} \tilde{\zeta}_{x_u x_v \dots x_w} \in A(Q)$$

and hence for the products

$$\zeta_{x_i x_j \dots x_k} \zeta_{x_u x_v \dots x_w} \in H^*(Q, \mathbf{Z}) .$$

Since such formulas do not seem to provide any additional insight we spare the reader and ourselves these added troubles.