

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COHOMOLOGY OF FINITELY GENERATED ABELIAN GROUPS
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Kapitel: 3. The proof of Theorem B
DOI: <https://doi.org/10.5169/seals-58730>

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3. THE PROOF OF THEOREM B

It is clear that the subring $C(Q)$ of the integral cohomology $H^*(Q, \mathbf{Z})$ generated by (the classes of) c_1, \dots, c_n has the defining relations

$$(3.1.1) \quad l_i c_i = 0, \quad 1 \leq i \leq n.$$

Since the c_i are Chern classes of the obvious 1-dimensional complex representations of Q we refer to $C(Q)$ as the *Chern ring* of Q .

THEOREM 3.1. *As a module over its Chern ring, the integral cohomology $H^*(Q, \mathbf{Z})$ of a finite abelian group $Q = C_{l_1} \times \dots \times C_{l_n}$ with $l_1 | l_2 | \dots | l_n$ is generated by 1 and the classes $\zeta_{x_i x_j \dots x_k}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least two, subject to the relations*

$$(3.1.2) \quad l_i \zeta_{x_i x_j \dots x_k} = 0.$$

Proof. We prove the Theorem by induction. It is clear that when Q is cyclic there is no monomial $\zeta_{x_i x_j \dots x_k}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least two and hence there is nothing to prove. Thus the induction starts.

Next, let

$$G = C_{l_1} \times \dots \times C_{l_n} = \langle t_1, \dots, t_n; t_j^{l_j} = 1 \rangle, \quad \text{with } l_1 | l_2 | \dots | l_n,$$

let

$$Q = G \times \mathbf{Z}/l = \langle t_1, \dots, t_n, t; t_j^{l_j} = 1, t^l = 1 \rangle,$$

and suppose that the exponent of G divides l , that is,

$$l_1 | l_2 | \dots | l_n | l.$$

It is manifest that the model $\mathcal{A}(Q)$ may be written

$$\mathcal{A}(Q) = \mathcal{A}(G) \otimes \mathcal{A}(\mathbf{Z}/l).$$

Regard the cycles $C(\mathcal{A}(\mathbf{Z}/l))$ and the boundaries $B(\mathcal{A}(\mathbf{Z}/l))$ as complexes with zero differential, and write $D(\mathcal{A}(\mathbf{Z}/l))$ for the boundaries $B(\mathcal{A}(\mathbf{Z}/l))$, regraded up by one, so that the exact sequence

$$0 \rightarrow C(\mathcal{A}(\mathbf{Z}/l)) \xrightarrow{\kappa} \mathcal{A}(\mathbf{Z}/l) \rightarrow D(\mathcal{A}(\mathbf{Z}/l)) \rightarrow 0$$

of chain complexes results. Since $\mathcal{A}(G)$ is free as a graded abelian group,

$$0 \rightarrow \mathcal{A}(G) \otimes C(\mathcal{A}(\mathbf{Z}/l)) \xrightarrow{\mathcal{A}(G) \otimes \kappa} \mathcal{A}(G) \otimes \mathcal{A}(\mathbf{Z}/l) \rightarrow \mathcal{A}(G) \otimes D(\mathcal{A}(\mathbf{Z}/l)) \rightarrow 0$$

is an exact sequence of chain complexes, too. In the standard way, cf. e.g. what is said on p. 166 of Mac Lane [23], its homology exact sequence boils down to the Künneth exact sequence

$$0 \rightarrow H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z}) \rightarrow H^*(Q, \mathbf{Z}) \rightarrow \text{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z})) \rightarrow 0 .$$

It is well known that this sequence splits. Exploiting the inductive hypothesis we conclude at once that, as a module over the Chern ring $C(Q)$, $H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \dots x_k}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least two, subject to the relations

$$l_i \zeta_{x_i x_j \dots x_k} = 0 .$$

Likewise, as a module over the Chern ring $C(Q)$, $\text{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ is generated by the images in $\text{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ of the classes $\zeta_{x_i x_j \dots x_k}$ of the kind (1.5) with $x_i x_j \dots x_k$ of degree at least one, subject to the relations

$$l_i \zeta_{x_i x_j \dots x_k} = 0 .$$

Since the Künneth sequence splits, this completes the proof. \square

We note that the above generators $\zeta_{x_i x_j \dots x_k}$ can presumably be understood in terms of the multi torsion product given in Mac Lane [25] generalizing the triple torsion product introduced in Mac Lane [24]. Details have not been worked out yet.

We now refer to the subring of $A(Q)$ generated by $\tilde{\zeta}_{x_1}, \tilde{\zeta}_{x_2}, \dots, \tilde{\zeta}_{x_n}$ as the *Chern ring* of $A(Q)$. It is clear that (1.13) identifies the Chern rings.

Proof of Theorem B. In view of (3.1), as a module over the Chern ring $C(Q)$, $H^*(Q, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \dots x_k}$ with $x_i x_j \dots x_k$ of degree at least two; hence (1.13) is an isomorphism over the Chern ring. Furthermore, (1.13) is induced by the restriction of (1.11) to the cycles in $\mathcal{A}(Q)$. Since the product structure in the cohomology ring is induced by the product structure in $\mathcal{A}(Q)$, and since the algebra $A(Q)$ arises from $\mathcal{A}(Q)$ by introducing the additional relations $l_i c_i = 0$, the association (1.13) identifies $H^*(Q, \mathbf{Z})$ with the subalgebra of $A(Q)$ generated by the $\tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$. \square

Under the circumstances of Theorem B it is straightforward to work out explicit formulas for the products

$$\tilde{\zeta}_{x_i x_j \dots x_k} \tilde{\zeta}_{x_u x_v \dots x_w} \in A(Q)$$

and hence for the products

$$\zeta_{x_i x_j \dots x_k} \zeta_{x_u x_v \dots x_w} \in H^*(Q, \mathbf{Z}) .$$

Since such formulas do not seem to provide any additional insight we spare the reader and ourselves these added troubles.