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COHOMOLOGY OF FINITELY GENERATED ABELIAN GROUPS

par Johannes HUEBSCHMANN

1. INTRODUCTION

Let Q be a finitely generated abelian group. In this note we compute its integral cohomology explicitly. Our approach is to construct a suitable small model $\mathscr{A}(Q)$ for the algebra of cochains on Q. Our model is very simple and relies on some differential homological algebra that has been created more than three decades ago [2-4], [8], [9], [28], [29]. Here is the model:

THEOREM A. *With coefficients in an arbitrary commutative R, the cohomology ring of a finitely generated abelian group Q, given by a presentation*

$$Q = < f_1, \ldots, f_m, t_1, \ldots, t_n \; ; \; t_j^{l_j} = 1 > \; ,$$

is as an R-algebra isomorphic to the homology algebra of the following primitively generated cocommutative differential graded Hopf algebra $\mathscr{A}(Q)$:

Generators: $z_1, \ldots, z_m, |z_j| = 1, x_1, \ldots, x_n, |x_i| = 1, c_1, \ldots, c_n, |c_i| = 2.$

Relations: *The generators commute in the graded sense, save that the squares of the x_i's are non-zero and given by*

$$x_i^2 = \frac{l_i(l_i - 1)}{2} c_i, 1 \leqslant i \leqslant n \; ; \tag{1.1}$$

in particular the squares of the z_j's are zero.

Differential: $d(z_j) = 0, 1 \leqslant j \leqslant m, \; d(x_i) = l_i c_i, \; d(c_i) = 0, 1 \leqslant i \leqslant n.$

For appropriate coefficients so that the diagonal map of this Hopf algebra induces a diagonal map on its homology, e.g. if R is a field or if the characteristic of R divides each $l_i, 1 \leqslant i \leqslant n,$ the isomorphism is one of Hopf algebras.

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Here as usual $|w|$ denotes the degree of a homogeneous element w. A proof of this Theorem has been given in Section 3 of our paper [13]. To make the present paper self contained we reproduce the proof in Section 2 below.

The model philosophy has already succesfully been exploited by us in [13-20] for theoretical purposes and to do calculations in group cohomology and algebraic topology. Models of different kinds have been introduced and exploited in the literature for decades, see for example [1-3], [9-11], [21], and the literature there; more about e.g. rational homotopy theory and minimal models may be found in Halperin's book [11].

Our main aim is to deduce an explicit description of the integral cohomology ring of Q from Theorem A. To explain our result, we denote an *infinite* cyclic group by C and a *finite* cyclic group of order l by C_l. We then write our abelian group Q in the form $Q = C_{l_1} \times \cdots \times C_{l_n} \times C^m$. It is clear that, as a graded commutative algebra, the integral cohomology ring $H^*(Q, \mathbf{Z})$ decomposes as

$$H^*(Q, \mathbf{Z}) = H^*(C_{l_1} \times \cdots \times C_{l_n}, \mathbf{Z}) \otimes H^*(C^m, \mathbf{Z}) \ . \tag{1.2}$$

Moreover, cf. Eckmann [7],

$$H^*(C^m, \mathbf{Z}) = \Lambda[z_1, \cdots, z_m] \tag{1.3}$$

where the notation z_j is abused somewhat. We therefore assume henceforth that Q is finite. Moreover, we suppose that things have been arranged in such a way that

$$l_1 \mid l_2 \mid \ldots \mid l_n \ , \tag{1.4}$$

where as usual '$u \mid v$' means that the number u divides the number v. It is classical that this is always possible. From Theorem A we deduce at once that every monomial of the kind

$$x_i x_j \ldots x_k, \qquad 1 \leqslant i < j < \cdots < k \leqslant n \ , \tag{1.5}$$

is a cycle in our algebra $\mathscr{A}(Q)$ when taken over the ground ring $R = \mathbf{Z}/l_i$ and hence determines a cohomology class $x_i x_j \ldots x_k \in H^*(-, \mathbf{Z}/l_i)$, where a slight abuse of notation comes into play. For each l_i, let

$$\beta_{l_i}\colon H^*(-, \mathbf{Z}/l_i) \to H^{*+1}(-, \mathbf{Z}) \tag{1.6}$$

be the indicated Bockstein operation and, for each monomial of the kind (1.5), let

$$\zeta_{x_i x_j \ldots x_k} = \beta_{l_i}(x_i x_j \ldots x_k) \in H^{*+1}(Q, \mathbf{Z}) \ . \tag{1.7}$$

It is clear that, in our model $\mathscr{A}(Q)$, the class $\zeta_{x_i x_j \dots x_k}$ is represented by the cocycle

(1.8) $$\frac{1}{l_i} d(x_i x_j \dots x_k) = c_i x_j \dots x_k - \frac{l_j}{l_i} c_j x_i \dots x_k + \cdots \pm \frac{l_k}{l_i} c_k x_i x_j \dots x_{k-1} .$$

Moreover, inspection of the Bockstein exact sequence

(1.9) $$H^*(Q,\mathbf{Z}) \xrightarrow{l_i} H^*(Q,\mathbf{Z}) \to H^*(Q,\mathbf{Z}/l_i) \xrightarrow{\beta_{l_i}} H^{*+1}(Q,\mathbf{Z})$$

shows that the class $\zeta_{x_i x_j \dots x_k} \in H^{*+1}(Q,\mathbf{Z})$ has exact order l_i.

We now consider the graded algebra $A(Q)$ that arises from $\mathscr{A}(Q)$ by introducing the additional relations $l_i c_i = 0$. It is clear that, when we write ζ_i for the obvious image of $c_i \in \mathscr{A}(Q)$ in $A(Q)$, as a graded algebra, $A(Q)$ is generated by

(1.10.1) $$x_1, \dots, x_n, |x_i| = 1 , \zeta_1, \dots, \zeta_n, |\zeta_i| = 2 ,$$

subject to the relations that

(1.10.2) the generators commute in the graded sense, save that the squares of the x_i's are possibly non-zero and given by

$$x_i^2 = \frac{l_i(l_i - 1)}{2} \zeta_i, 1 \leqslant i \leqslant n ;$$

and

(1.10.3) $$l_i \zeta_i = 0, 1 \leqslant i \leqslant n .$$

We note that when l_i is odd we have in fact $x_i^2 = 0$. By construction, the association

$$c_i \mapsto \zeta_i , \quad x_j \mapsto x_j$$

yields a surjective morphism

(1.11) $$\mathscr{A}(Q) \to A(Q)$$

of differential graded algebras where the algebra $A(Q)$ is understood to have zero differential. Since (1.11) is an isomorphism in degree 1 we do not distinguish in notation between $x_j \in \mathscr{A}(Q)$ and its image in $A(Q)$. For each monomial $x_i x_j \dots x_k$ of the kind (1.5), we write

$$\tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$$

for the image of $\zeta_{x_i x_j \dots x_k} \in \mathscr{A}(Q)$ under (1.11), so that

$$(1.12)\quad \tilde{\zeta}_{x_i x_j \dots x_k} = \zeta_i x_j \dots x_k - \frac{l_j}{l_i}\zeta_j x_i \dots x_k + \cdots \pm \frac{l_k}{l_i}\zeta_k x_i x_j \dots x_{k-1} \in A(Q)\,.$$

Here is our main result.

THEOREM B. *For a finite abelian group Q, the association*

$$(1.13)\qquad \zeta_{x_i x_j \dots x_k} \mapsto \tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$$

identifies $H^*(Q, \mathbf{Z})$ with the graded subalgebra of $A(Q)$ generated (as an algebra) by the $\tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$.

This description of the cohomology ring should be compared with that in Chapman [5]. Our description of the cohomology ring has its advantages and disadvantages. An advantage is that it arises from a "small" model; indeed, modulo a prime p dividing each torsion coefficient l_i, our model boils down to the cohomology ring itself. To have a model as small as possible is important for explicit computations. A disadvantage of our description is that it is natural in the *presentation only,* and not in the group itself. Invariant descriptions of the *homology* of a finitely generated abelian group have been given by Hamsher [12] and Decker [6] in their Chicago ph. d. theses supervised by S. Mac Lane. We do not known whether invariant descriptions of the *cohomology* of a finitely generated abelian group have ever been worked out.

I am indebted to S. Mac Lane for discussions and for a number of comments about an earlier version of the paper.

2. THE PROOF OF THEOREM A

To make the paper self contained we reproduce the following material from our paper [13]. The contents of the present Section are of course classical but apparently not as well known as they deserve.

Let R be a commutative ring with 1. By a *Hopf algebra* $(H, \mu, \Delta, \eta, \varepsilon)$ over R we mean as usual a module H together with the structures (μ, η) and (Δ, ε) of an algebra and a coalgebra that are compatible, that is, μ is a morphism of coalgebras or, equivalently, Δ is a morphism of algebras; see e.g. VI.9 in Mac Lane [23]. We mention in passing that some authors call this a bialgebra and require a Hopf algebra to have the additional structure of what is called an *antipode*. For us this is actually of no account since this additional structure will always be present, but there is no need to spell it out. Examples of Hopf algebras are group rings, exterior Hopf algebras, divided polynomial Hopf algebras, polynomial Hopf algebras, see e.g. Mac Lane [23]. Henceforth we write $\Lambda[\upsilon]$ for the exterior Hopf algebra on a single generator υ. As a coalgebra, the divided polynomial Hopf algebra is the symmetric coalgebra over its primitives. For the reader's convenience, and to introduce notation, we recall that, for $i \geqslant 1$, the *divided polynomial Hopf algebra* $\Gamma = \Gamma[u]$ *on* a single generator u of even degree has a generator $\gamma_i(u)$ of degree $|\gamma_i(u)| = i|u|$, subject to the relations

$$\gamma_i(u)\gamma_j(u) = \binom{i+j}{j}\gamma_{i+j}(u), \qquad i, j \geqslant 1 .$$

Furthermore, the rule

$$\Delta\gamma_i(u) = \sum_{j+k=i} (\gamma_j(u)) \otimes (\gamma_k(u)), \qquad i \geqslant 1 ,$$

yields a diagonal map Δ on Γ which endows the latter with the structure of a graded Hopf algebra. The span M of u in $\Gamma[u]$ is the module of *primitives*, and, as a graded coalgebra, Γ is the *symmetric* coalgebra over M. Notice that if R has characteristic zero, the diagonal map Δ is uniquely determined by the requirements that (i) Γ is a Hopf algebra, and that (ii) u primitive.

Let C be an *infinite* cyclic group, pick a generator y, and let $\Lambda[\upsilon_y]$ be the exterior Hopf algebra on a single element υ_y of degree 1. We mention that υ_y may be identified with the suspension of $y - 1 \in RC$, cf. Section 16 of Eilenberg-Mac Lane [9.I], exposé 6 of Cartan [3], but we shall not need this fact. It is well known that the standard small free resolution of R in the category of right RC-modules may be written as a differential graded Hopf algebra

$$M(C) = (\Lambda[\upsilon_y] \otimes RC, d, \mu, \Delta, \varepsilon, \eta) = (M^{\#}(C) \otimes_d RC, \mu, \Delta, \varepsilon, \eta) . \tag{2.1}$$

Here as a graded commutative algebra, $M^{\#}(C) = \Lambda[\upsilon_y]$, and the underlying graded commutative algebra of $M(C)$ has the tensor product structure, that is, it looks like $M^{\#}(C) \otimes RC$; henceforth we shall discard the tensor product symbol and write $\upsilon_y \otimes 1 = \upsilon_y$ etc. Furthermore, the other structure maps d, Δ, ε, and η are given by the well known formulas

$$\begin{gathered} d(\upsilon_y) = (y-1) , \quad \varepsilon(y^k) = 1 \in R, k \geqslant 0 , \quad \eta(1) = 1 \in RC , \\ \Delta(y) = y \otimes y , \quad \Delta(\upsilon_y) = \upsilon_y \otimes y + 1 \otimes \upsilon_y . \end{gathered} \tag{2.2}$$

Notice that the diagonal map Δ is *not* cocommutative.

Likewise, for a *finite* cyclic group $C_l = \langle y; y^l = 1\rangle$, the standard small free resolution of R in the category of right RC_l-modules may be written as an augmented and coaugmented differential graded algebra with diagonal (in the sense of Section 1 of [13])

$$\begin{aligned} M(C_l) &= (\Gamma[u_y] \otimes \Lambda[v_y] \otimes RC_l, d, \mu, \Delta, \varepsilon, \eta) \\ &= (M^{\#}(C_l) \otimes_d RC_l, \mu, \Delta, \varepsilon, \eta)\,. \end{aligned} \tag{2.3}$$

Here as a graded commutative algebra, $M^{\#}(C_l) = \Gamma[u_y] \otimes \Lambda[v_y]$ with the tensor product structure, and the underlying graded commutative algebra of $M(C_l)$ has the tensor product structure, too, that is, it looks like $M^{\#}(C_l) \otimes RC_l$; as above we shall henceforth discard the tensor product symbol and write $u_y \otimes 1 = u_y$ etc. Furthermore, the other structure maps d, Δ, ε, and η are given by the well known formulas

$$\begin{aligned} d(\gamma_j(u_y)) &= (\gamma_{j-1}(u_y))v_y(1 + y + \cdots + y^{l-1}), \\ d((\gamma_j(u_y))v_y) &= (\gamma_j(u_y))(y-1), \\ \Delta(y) &= y \otimes y\,, \\ \Delta(\gamma_j(u_y)) &= \sum_{0 \leqslant i \leqslant j} \Bigg(\gamma_i(u_y) \otimes \gamma_{j-i}(u_y) \\ &\quad + \sum_{0 \leqslant m < n \leqslant l-1} (\gamma_i(u_y))\, v_y y^m \otimes (\gamma_{j-i-1}(u_y))\, v_y y^n \Bigg) \qquad (2.4) \\ \Delta((\gamma_j(u_y))v_y) &= \sum_{0 \leqslant i \leqslant j} ((\gamma_i(u_y))v_y \otimes (\gamma_{j-i}(u_y))y + (\gamma_i(u_y)) \otimes (\gamma_{j-i}(u_y))v_y)\,, \\ \varepsilon(y^k) &= 1 \in R,\ 0 \leqslant k < l\,, \qquad \eta(1) = 1 \in RC_l\,, \end{aligned}$$

where $j \geqslant 0$ or $j \geqslant 1$ as appropriate. Notice that the diagonal map Δ is no longer coassociative. We indicated in [13] (3.2.14) and (3.4.4) how these formulas can be obtained from scratch.

We now write $\bar{M}(C) = M(C) \otimes_{RC} R$ and $\bar{M}(C_l) = M(C_l) \otimes_{RC_l} R$; these are the corresponding *reduced objects* [13]. It is clear that, as differential graded commutative algebras, they look like

(2.5) $\bar{M}(C) = \Lambda[v_y]$ with zero differential,

$$\bar{M}(C_l) = (\Gamma[u_y] \otimes \Lambda[v_y], d), \quad \text{where} \begin{cases} d(\gamma_j(u_y)) = l(\gamma_{j-1}(u_y))v_y\,, \\ d((\gamma_j(u_y))v_y) = 0\,. \end{cases} \tag{2.6}$$

Furthermore, it is manifest that $\bar{M}(C) = \Lambda[v_y] = H_*(C, R)$ as Hopf algebras, and that the above diagonal map induces a diagonal map Δ on $\bar{M}(C_l)$ given by

$$\Delta(\gamma_j(u_y)) = \sum_{0 \leqslant i \leqslant j} \Bigg(\gamma_i(u_y) \otimes \gamma_{j-i}(u_y)$$

$$(2.7)\qquad + \frac{l(l-1)}{2}(\gamma_i(u_y))v_y \otimes (\gamma_{j-i-1}(u_y))v_y\Big) ,$$

$$\Delta((\gamma_j(u_y))v_y) = \sum_{0 \leqslant i \leqslant j} ((\gamma_i(u_y))v_y \otimes (\gamma_{j-i}(u_y)) + (\gamma_i(u_y)) \otimes (\gamma_{j-i}(u_y))v_y) .$$

Inspection shows that the latter induces in fact the structure of a (non-cocommutative) differential graded Hopf algebra on $\bar{M}(C_l)$.

If $Q = C_{l_1} \times \cdots \times C_{l_n} \times C^m$ is a finitely generated abelian group, written out as a direct product of cyclic groups as indicated, it is clear that, with the obvious action, the object

$$(2.8)\qquad M(Q) = M(C_{l_1}) \otimes \cdots \otimes M(C_{l_n}) \otimes (M(C))^{\otimes m}$$

is a free resolution of R in the category of RQ-modules. Inspection shows that it coincides with the corresponding free resolution introduced by Tate [29]. Moreover, the tensor product structures turn $M(Q)$ into an augmented and coaugmented differential graded commutative algebra with diagonal. Likewise, the reduced object $\bar{M}(Q) = M(Q) \otimes_{RQ} R$ looks like

$$(2.9)\qquad \bar{M}(Q) = \bar{M}(C_{l_1}) \otimes \cdots \otimes \bar{M}(C_{l_n}) \otimes (\bar{M}(C))^{\otimes m} ,$$

and the tensor product structures turn it into a differential graded commutative Hopf algebra. From this one deduces at once a proof of Theorem A.

Remark 2.10. A few historical comments seem in order: The standard small free resolution of a finite cyclic group is due to Eilenberg and Mac Lane and was first published in § 11 of Eilenberg [8]. The reduced objects $\bar{M}(C)$ and $\bar{M}(C_l)$ without the diagonal map were introduced in Ch. III of Eilenberg-Mac Lane [9.II]; furthermore, in the same reference it is proved that for any finitely generated abelian group Q written out as a direct product as above, as a differential graded algebra, the object $\bar{M}(Q)$ is chain equivalent to the reduced bar construction on the group ring RQ. Moreover, I learnt from Mac Lane that in 1952 he and Eilenberg were aware of the diagonal map spelled out above. The resolution for a finite cyclic group with all the above structure appears on p. 252 of Cartan-Eilenberg [4]; there the resolution is written X_L and the names divided polynomial algebra or Hopf algebra do not occur but the structure is given explicitly. Furthermore, the objects $M(C)$, $M(C_l)$, and $\bar{M}(C_l)$ were recognised as *constructions* by Cartan [2], see also [3] and §§ 2 and 3 of Moore [28]; this fact is heavily exploited in our paper [13].

3. The proof of Theorem B

It is clear that the subring $C(Q)$ of the integral cohomology $H^*(Q, \mathbf{Z})$ generated by (the classes of) $c_1, \ldots, c_n$ has the defining relations

$$l_i c_i = 0, \quad 1 \leqslant i \leqslant n. \tag{3.1.1}$$

Since the c_i are Chern classes of the obvious 1-dimensional complex representations of Q we refer to $C(Q)$ as the *Chern ring* of Q.

THEOREM 3.1. *As a module over its Chern ring, the integral cohomology* $H^*(Q, \mathbf{Z})$ *of a finite abelian group* $Q = C_{l_1} \times \cdots \times C_{l_n}$ *with* $l_1 \mid l_2 \mid \ldots \mid l_n$ *is generated by* 1 *and the classes* $\zeta_{x_i x_j \ldots x_k}$ *of the kind (1.5) with* $x_i x_j \ldots x_k$ *of degree at least two, subject to the relations*

$$l_i \zeta_{x_i x_j \ldots x_k} = 0. \tag{3.1.2}$$

Proof. We prove the Theorem by induction. It is clear that when Q is cyclic there is no monomial $\zeta_{x_i x_j \ldots x_k}$ of the kind (1.5) with $x_i x_j \ldots x_k$ of degree at least two and hence there is nothing to prove. Thus the induction starts.

Next, let

$$G = C_{l_1} \times \cdots \times C_{l_n} = < t_1, \ldots, t_n; \quad t_j^{l_j} = 1 >, \quad \text{with} \quad l_1 \mid l_2 \mid \ldots \mid l_n,$$

let

$$Q = G \times \mathbf{Z}/l = < t_1, \ldots, t_n, t; t_j^{l_j} = 1, t^l = 1 >,$$

and suppose that the exponent of G divides l, that is,

$$l_1 \mid l_2 \mid \ldots \mid l_n \mid l.$$

It is manifest that the model $\mathscr{A}(Q)$ may be written

$$\mathscr{A}(Q) = \mathscr{A}(G) \otimes \mathscr{A}(\mathbf{Z}/l).$$

Regard the cycles $C(\mathscr{A}(\mathbf{Z}/l))$ and the boundaries $B(\mathscr{A}(\mathbf{Z}/l))$ as complexes with zero differential, and write $D(\mathscr{A}(\mathbf{Z}/l))$ for the boundaries $B(\mathscr{A}(\mathbf{Z}/l))$, regraded up by one, so that the exact sequence

$$0 \to C(\mathscr{A}(\mathbf{Z}/l)) \xrightarrow{\kappa} \mathscr{A}(\mathbf{Z}/l) \to D(\mathscr{A}(\mathbf{Z}/l)) \to 0$$

of chain complexes results. Since $\mathscr{A}(G)$ is free as a graded abelian group,

$$0 \to \mathscr{A}(G) \otimes C(\mathscr{A}(\mathbf{Z}/l)) \xrightarrow{\mathscr{A}(G) \otimes \kappa} \mathscr{A}(G) \otimes \mathscr{A}(\mathbf{Z}/l) \to \mathscr{A}(G) \otimes D(\mathscr{A}(\mathbf{Z}/l)) \to 0$$

is an exact sequence of chain complexes, too. In the standard way, cf. e.g. what is said on p. 166 of Mac Lane [23], its homology exact sequence boils down to the Künneth exact sequence

$$0 \to H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z}) \to H^*(Q, \mathbf{Z}) \to \operatorname{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z})) \to 0 \ .$$

It is well known that this sequence splits. Exploiting the inductive hypothesis we conclude at once that, as a module over the Chern ring $C(Q)$, $H^*(G, \mathbf{Z}) \otimes H^*(\mathbf{Z}/l, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \ldots x_k}$ of the kind (1.5) with $x_i x_j \ldots x_k$ of degree at least two, subject to the relations

$$l_i \zeta_{x_i x_j \ldots x_k} = 0 \ .$$

Likewise, as a module over the Chern ring $C(Q)$, $\operatorname{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ is generated by the images in $\operatorname{Tor}(H^*(G, \mathbf{Z}), H^*(\mathbf{Z}/l, \mathbf{Z}))$ of the classes $\zeta_{x_i x_j \ldots x_k x}$ of the kind (1.5) with $x_i x_j \ldots x_k$ of degree at least one, subject to the relations

$$l_i \zeta_{x_i x_j \ldots x_k} = 0 \ .$$

Since the Künneth sequence splits, this completes the proof. □

We note that the above generators $\zeta_{x_i x_j \ldots x_k}$ can presumably be understood in terms of the multi torsion product given in Mac Lane [25] generalizing the triple torsion product introduced in Mac Lane [24]. Details have not been worked out yet.

We now refer to the subring of $A(Q)$ generated by $\tilde{\zeta}_{x_1}, \tilde{\zeta}_{x_2}, \ldots, \tilde{\zeta}_{x_n}$ as the *Chern ring* of $A(Q)$. It is clear that (1.13) identifies the Chern rings.

Proof of Theorem B. In view of (3.1), as a module over the Chern ring $C(Q)$, $H^*(Q, \mathbf{Z})$ is generated by 1 and the classes $\zeta_{x_i x_j \ldots x_k}$ with $x_i x_j \ldots x_k$ of degree at least two; hence (1.13) is an isomorphism over the Chern ring. Furthermore, (1.13) is induced by the restriction of (1.11) to the cycles in $\mathscr{A}(Q)$. Since the product structure in the cohomology ring is induced by the product structure in $\mathscr{A}(Q)$, and since the algebra $A(Q)$ arises from $\mathscr{A}(Q)$ by introducing the additional relations $l_i c_i = 0$, the association (1.13) identifies $H^*(Q, \mathbf{Z})$ with the subalgebra of $A(Q)$ generated by the $\tilde{\zeta}_{x_i x_j \ldots x_k} \in A(Q)$. □

Under the circumstances of Theorem B it is straightforward to work out explicit formulas for the products

$$\tilde{\zeta}_{x_i x_j \ldots x_k} \tilde{\zeta}_{x_u x_v \ldots x_w} \in A(Q)$$

and hence for the products

$$\zeta_{x_i x_j \ldots x_k} \zeta_{x_u x_v \ldots x_w} \in H^*(Q, \mathbf{Z}) \ .$$

Since such formulas do not seem to provide any additional insight we spare the reader and ourselves these added troubles.

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