5. Spectral decomposition

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we get

$$-\frac{1}{2\pi}\int_0^{\pi}\psi_1(\theta)\psi_2(\theta)\left((q-1)^2+4q\sin^2(\theta)\right)d\theta\ ,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. Spectral decomposition

Let E be the space of functions F_ψ with $\psi \in L^2([0,\pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T. Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E\ .$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $< g, F_\psi > = 0$ for all ψ, i.e.,

$$0 = \frac{1}{q+1}g(0)\frac{1}{2\pi}\int_{-\pi}^{\pi} i(q+1)\sin\theta\psi(\theta)d\theta$$

$$+\sum_{n=1}^{\infty} g(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(\theta)i(\sin((n+1)\theta)-q\sin((n-1)\theta))d\theta q^{-\frac{n}{2}}$$

$$= g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)(\hat{\psi}(n+1)-q\hat{\psi}(n-1))q^{-\frac{n}{2}}\ .$$

Therefore

$$g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)\hat{\psi}(n+1)q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ ,$$

or (as $\hat{\psi}(0) = 0$)

$$\sum_{n=1}^{\infty} g(n-1)\hat{\psi}(n)q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ .$$

Since $\psi \in L^2([0,\pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over

$L^2([0,\pi])$, the sequences $\{\hat{\psi}(n)\}$ vary over l^2. Since the latter is a Hilbert space, it follows that for all n,

$$g(n-1) = g(n+1)\ .$$

Let $a = \frac{1}{2}(g(0)+g(1))$, $b = \frac{1}{2}(g(0)-g(1))$. Then

$$g(n) = a + (-1)^n b\ ,$$

a typical member of R.

COROLLARY 5.2. *The spectrum of* T *on* $L^2(F)$ *is the subset of* **R** *described in the Introduction.*

We wish to make this decomposition explicit, that is, given $g \in E$ to find the ψ such that $g = F_\psi$ (compare [H]). Let $\phi(\theta)$ be the characteristic function of $[\theta_0, \theta_0 + h] \subset [0,\pi]$. Then

$$F_\phi(n) = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n)\,d\theta\ .$$

By the Plancherel formula (6),

$$<g, F_\phi> \;=\; <g, \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n)\,d\theta>$$

$$= \frac{1}{2\pi}\int_0^\pi \psi(\theta)\phi(\theta)\left((q-1)^2 + 4q\sin^2\theta\right)d\theta$$

$$= \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+h} \psi(\theta)\left((q-1)^2 + 4q\sin^2\theta\right)d\theta\ .$$

Supposing further that g belongs in the dense subspace $E \cap L^1(F)$, we divide by h and let $h \to 0$ to obtain

$$<g, \tilde{f}_{\theta_0}> \;=\; \psi(\theta_0)\left((q-1)^2 + 4q\sin^2\theta\right)/2\pi\ .$$

Hence:

THEOREM 5.3. *Let* u_1 *and* u_2 *be the orthonormal basis of* R *given by*

$$u_1(n) \equiv \sqrt{\frac{q^2-1}{2q}}\,,\qquad u_2(n) = \sqrt{\frac{q^2-1}{2q}}\,(-1)^n\ .$$

Then the spectral resolution of $L^2(F)$ *reads*

(7)

$$g(n) = \sum_{i=1,2} < g, u_i > u_i(n) + 2\pi \int_0^{\pi} < g, \tilde{f}_\theta > \tilde{f}_\theta(n) \frac{d\theta}{(q-1)^2 + 4q\sin^2\theta} .$$

We end this paper by showing that, as one might expect from the theory of Eisenstein series, the eigenfunctions f_λ can be parametrized as a family of functions that depend holomorphically on a complex parameter. Precisely, let

$$E(n,s) = q^{ns}(q^{s-1} - q^{1-s}) + q^{n(1-s)}(q^s - q^{-s}) .$$

Then $E(n,s)$ is entire in s and satisfies the functional equation

$$E(n,s) = -E(n, 1-s) .$$

Furthermore, a direct computation shows that

$$(TE)(n,s) = (q^s + q^{1-s})E(n,s) .$$

There are two ways in which $\lambda = q^s + q^{1-s}$ can be real. Write $s = \sigma + it$. If $t = \dfrac{k\pi}{\log q}$ then $\lambda = (-1)^k(q^\sigma + q^{1-\sigma})$, and in particular

$$\lambda = q + 1 (\lambda = -(q+1))$$

if $\sigma = 1$ and k is even (k is odd). Otherwise we must have $\sigma = \dfrac{1}{2}$. If we write $t = \dfrac{\theta}{\log q}$ we obtain our $\lambda = 2\sqrt{q}\cos\theta$.

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