

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AUTOMORPHIC SPECTRA ON THE TREE OF PSL_2
Autor: Efrat, Isaac
Kapitel: 5. Spectral decomposition
DOI: <https://doi.org/10.5169/seals-58728>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

we get

$$-\frac{1}{2\pi} \int_0^\pi \psi_1(\theta) \psi_2(\theta) ((q-1)^2 + 4q \sin^2(\theta)) d\theta,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. SPECTRAL DECOMPOSITION

Let E be the space of functions F_ψ with $\psi \in L^2([0, \pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T . Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E.$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $\langle g, F_\psi \rangle = 0$ for all ψ , i.e.,

$$\begin{aligned} 0 &= \frac{1}{q+1} g(0) \frac{1}{2\pi} \int_{-\pi}^\pi i(q+1) \sin \theta \psi(\theta) d\theta \\ &+ \sum_{n=1}^{\infty} g(n) \frac{1}{2\pi} \int_{-\pi}^\pi \psi(\theta) i(\sin((n+1)\theta) - q \sin((n-1)\theta)) d\theta q^{-\frac{n}{2}} \\ &= g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) (\hat{\psi}(n+1) - q \hat{\psi}(n-1)) q^{-\frac{n}{2}}. \end{aligned}$$

Therefore

$$g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) \hat{\psi}(n+1) q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}},$$

or (as $\hat{\psi}(0) = 0$)

$$\sum_{n=1}^{\infty} g(n-1) \hat{\psi}(n) q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}}.$$

Since $\psi \in L^2([0, \pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over

$L^2([0, \pi])$, the sequences $\{\hat{\psi}(n)\}$ vary over l^2 . Since the latter is a Hilbert space, it follows that for all n ,

$$g(n - 1) = g(n + 1) .$$

Let $a = \frac{1}{2} (g(0) + g(1))$, $b = \frac{1}{2} (g(0) - g(1))$. Then

$$g(n) = a + (-1)^n b ,$$

a typical member of R .

COROLLARY 5.2. *The spectrum of T on $L^2(F)$ is the subset of \mathbf{R} described in the Introduction.*

We wish to make this decomposition explicit, that is, given $g \in E$ to find the ψ such that $g = F_\psi$ (compare [H]). Let $\phi(\theta)$ be the characteristic function of $[\theta_0, \theta_0 + h] \subset [0, \pi]$. Then

$$F_\phi(n) = \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n) d\theta .$$

By the Plancherel formula (6),

$$\begin{aligned} \langle g, F_\phi \rangle &= \langle g, \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n) d\theta \rangle \\ &= \frac{1}{2\pi} \int_0^\pi \psi(\theta) \phi(\theta) ((q - 1)^2 + 4q \sin^2 \theta) d\theta \\ &= \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \psi(\theta) ((q - 1)^2 + 4q \sin^2 \theta) d\theta . \end{aligned}$$

Supposing further that g belongs in the dense subspace $E \cap L^1(F)$, we divide by h and let $h \rightarrow 0$ to obtain

$$\langle g, \tilde{f}_{\theta_0} \rangle = \psi(\theta_0) ((q - 1)^2 + 4q \sin^2 \theta) / 2\pi .$$

Hence:

THEOREM 5.3. *Let u_1 and u_2 be the orthonormal basis of R given by*

$$u_1(n) \equiv \sqrt{\frac{q^2 - 1}{2q}} , \quad u_2(n) = \sqrt{\frac{q^2 - 1}{2q}} (-1)^n .$$

Then the spectral resolution of $L^2(F)$ reads

(7)

$$g(n) = \sum_{i=1,2} \langle g, u_i \rangle u_i(n) + 2\pi \int_0^\pi \langle g, \tilde{f}_\theta \rangle \tilde{f}_\theta(n) \frac{d\theta}{(q-1)^2 + 4q \sin^2 \theta}.$$

We end this paper by showing that, as one might expect from the theory of Eisenstein series, the eigenfunctions f_λ can be parametrized as a family of functions that depend holomorphically on a complex parameter. Precisely, let

$$E(n, s) = q^{ns}(q^{s-1} - q^{1-s}) + q^{n(1-s)}(q^s - q^{-s}).$$

Then $E(n, s)$ is entire in s and satisfies the functional equation

$$E(n, s) = -E(n, 1-s).$$

Furthermore, a direct computation shows that

$$(TE)(n, s) = (q^s + q^{1-s})E(n, s).$$

There are two ways in which $\lambda = q^s + q^{1-s}$ can be real. Write $s = \sigma + it$.

If $t = \frac{k\pi}{\log q}$ then $\lambda = (-1)^k(q^\sigma + q^{1-\sigma})$, and in particular

$$\lambda = q + 1 \quad (\lambda = -(q + 1))$$

if $\sigma = 1$ and k is even (k is odd). Otherwise we must have $\sigma = \frac{1}{2}$. If we write

$t = \frac{\theta}{\log q}$ we obtain our $\lambda = 2\sqrt{q} \cos \theta$.

REFERENCES

- [B] BIGGS, N. *Algebraic Graph Theory*. Cambridge University Press, 1974.
- [C1] CARTIER, P. Géométrie et analyse sur les arbres. *Sém. Bourbaki 1971/2*, exposé 407.
- [C2] ——— Harmonic analysis on trees. *Proc. of Symp. on Pure Mathematics*, vol. XXVI, 1973.
- [D] DRINFELD, V. G. Number of two-dimensional irreducible representations of the fundamental group of a curve over a finite field. *Funct. Anal. Appl.* 15, 4 (1982), 294-295.
- [E0] EFRAT, I. Automorphic spectra on the tree of PGL_2 . *Publ. of MSRI*, No. 08908 (1986).