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we get

$$-\frac{1}{2\pi} \int_0^\pi \psi_1(\theta) \psi_2(\theta) ((q-1)^2 + 4q \sin^2(\theta)) d\theta,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. SPECTRAL DECOMPOSITION

Let E be the space of functions F_ψ with $\psi \in L^2([0, \pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T . Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E.$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $\langle g, F_\psi \rangle = 0$ for all ψ , i.e.,

$$\begin{aligned} 0 &= \frac{1}{q+1} g(0) \frac{1}{2\pi} \int_{-\pi}^\pi i(q+1) \sin \theta \psi(\theta) d\theta \\ &+ \sum_{n=1}^{\infty} g(n) \frac{1}{2\pi} \int_{-\pi}^\pi \psi(\theta) i(\sin((n+1)\theta) - q \sin((n-1)\theta)) d\theta q^{-\frac{n}{2}} \\ &= g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) (\hat{\psi}(n+1) - q \hat{\psi}(n-1)) q^{-\frac{n}{2}}. \end{aligned}$$

Therefore

$$g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) \hat{\psi}(n+1) q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}},$$

or (as $\hat{\psi}(0) = 0$)

$$\sum_{n=1}^{\infty} g(n-1) \hat{\psi}(n) q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}}.$$

Since $\psi \in L^2([0, \pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over

$L^2([0, \pi])$, the sequences $\{\hat{\psi}(n)\}$ vary over l^2 . Since the latter is a Hilbert space, it follows that for all n ,

$$g(n-1) = g(n+1) .$$

Let $a = \frac{1}{2}(g(0) + g(1))$, $b = \frac{1}{2}(g(0) - g(1))$. Then

$$g(n) = a + (-1)^n b ,$$

a typical member of R .

COROLLARY 5.2. *The spectrum of T on $L^2(F)$ is the subset of \mathbf{R} described in the Introduction.*

We wish to make this decomposition explicit, that is, given $g \in E$ to find the ψ such that $g = F_\psi$ (compare [H]). Let $\phi(\theta)$ be the characteristic function of $[\theta_0, \theta_0 + h] \subset [0, \pi]$. Then

$$F_\phi(n) = \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n) d\theta .$$

By the Plancherel formula (6),

$$\begin{aligned} \langle g, F_\phi \rangle &= \langle g, \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n) d\theta \rangle \\ &= \frac{1}{2\pi} \int_0^\pi \psi(\theta) \phi(\theta) ((q-1)^2 + 4q \sin^2 \theta) d\theta \\ &= \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \psi(\theta) ((q-1)^2 + 4q \sin^2 \theta) d\theta . \end{aligned}$$

Supposing further that g belongs in the dense subspace $E \cap L^1(F)$, we divide by h and let $h \rightarrow 0$ to obtain

$$\langle g, \tilde{f}_{\theta_0} \rangle = \psi(\theta_0) ((q-1)^2 + 4q \sin^2 \theta) / 2\pi .$$

Hence:

THEOREM 5.3. *Let u_1 and u_2 be the orthonormal basis of R given by*

$$u_1(n) \equiv \sqrt{\frac{q^2-1}{2q}} , \quad u_2(n) = \sqrt{\frac{q^2-1}{2q}} (-1)^n .$$

Then the spectral resolution of $L^2(F)$ reads

(7)

$$g(n) = \sum_{i=1,2} \langle g, u_i \rangle u_i(n) + 2\pi \int_0^\pi \langle g, \tilde{f}_\theta \rangle \tilde{f}_\theta(n) \frac{d\theta}{(q-1)^2 + 4q \sin^2 \theta}.$$

We end this paper by showing that, as one might expect from the theory of Eisenstein series, the eigenfunctions f_λ can be parametrized as a family of functions that depend holomorphically on a complex parameter. Precisely, let

$$E(n, s) = q^{ns}(q^{s-1} - q^{1-s}) + q^{n(1-s)}(q^s - q^{-s}).$$

Then $E(n, s)$ is entire in s and satisfies the functional equation

$$E(n, s) = -E(n, 1-s).$$

Furthermore, a direct computation shows that

$$(TE)(n, s) = (q^s + q^{1-s})E(n, s).$$

There are two ways in which $\lambda = q^s + q^{1-s}$ can be real. Write $s = \sigma + it$.

If $t = \frac{k\pi}{\log q}$ then $\lambda = (-1)^k(q^\sigma + q^{1-\sigma})$, and in particular

$$\lambda = q + 1 \quad (\lambda = -(q + 1))$$

if $\sigma = 1$ and k is even (k is odd). Otherwise we must have $\sigma = \frac{1}{2}$. If we write

$t = \frac{\theta}{\log q}$ we obtain our $\lambda = 2\sqrt{q} \cos \theta$.

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