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Then, for $n \geq 1$,

$$(5) \quad \begin{aligned} \tilde{f}_\theta(n) &= q^{\frac{n}{2}} i((q+1) \sin \theta \cos(n\theta) - (q-1) \cos \theta \sin(n\theta)) \\ &= q^{\frac{n}{2}} i(\sin((n+1)\theta) - q \sin((n-1)\theta)), \end{aligned}$$

and

$$\tilde{f}_\theta(0) = (q+1) i \sin \theta .$$

PROPOSITION 3.6. *The functions $\tilde{f}_\theta, 0 < \theta < \pi$, are not in $L^2(F)$.*

Proof. It is sufficient to show that

$$(q+1) \sin \theta \cos(n\theta) - (q-1) \cos \theta \sin(n\theta) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

This is the dot product of the two vectors

$$v_1 = ((q+1) \sin \theta, -(q-1) \cos \theta) \quad \text{and} \quad v_2 = (\cos(n\theta), \sin(n\theta)) .$$

Since v_2 is not a constant function of n , we see that cosine of the angle between v_1 and v_2 is bounded away from 0 for arbitrarily large n .

Combining Propositions 3.5 and 3.6 we conclude

COROLLARY 3.7. *The discrete spectrum of T consists of the numbers $\pm(q+1)$, whose corresponding eigenfunctions (given in Example 3.1) span two one-dimensional eigenspaces of $L^2(F)$.*

Unlike the typical f_λ with $|\lambda| > 2\sqrt{q}$, those with $|\lambda| < 2\sqrt{q}$ satisfy

$$f_\lambda = O(q^{\frac{n}{2}}) .$$

Our goal now is to show that these are approximate eigenfunctions that can be used to completely decompose $L^2(F)$.

4. CONTINUOUS SPECTRA

We wish to embed $L^2([0, \pi])$ with an appropriate measure into $L^2(F)$. To this end, let $\psi \in L^2([0, \pi])$ and $\tilde{f}_\theta(n)$ be extended as odd functions of $\theta \in [-\pi, \pi]$, and define

$$F_\psi(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\theta) \tilde{f}_\theta(n) d\theta .$$

THEOREM 4.1. F_ψ is in $L^2(F)$ and we have the Plancherel formula

$$(6) \quad \langle F_{\psi_1}, F_{\psi_2} \rangle = \langle \psi_1, \psi_2 \rangle ,$$

where the inner product on the right is

$$\frac{1}{2\pi} \int_0^\pi \psi_1(\theta) \bar{\psi}_2(\theta) ((q-1)^2 + 4q \sin^2 \theta) d\theta .$$

Proof. We first note that

$$\frac{1}{2\pi} \int_\pi^\pi \psi(\theta) (i \sin((n+1)\theta) - q i \sin((n-1)\theta)) d\theta = q \hat{\psi}(n-1) - \hat{\psi}(n+1) ,$$

where $\hat{\psi}(n)$ is the n -th Fourier coefficient of ψ . Therefore

$$\begin{aligned} \langle F_{\psi_1}, \bar{F}_{\psi_2} \rangle &= \frac{1}{q+1} (q+1)^2 \hat{\psi}_1(1) \hat{\psi}_2(1) \\ &+ \sum_{n=1}^{\infty} (q^2 \hat{\psi}_1(n-1) \hat{\psi}_2(n-1) - q \hat{\psi}_1(n+1) \hat{\psi}_2(n-1) \\ &\quad - q \hat{\psi}_1(n-1) \hat{\psi}_2(n+1) + \hat{\psi}_1(n+1) \hat{\psi}_2(n+1)) \\ &= (q+1) \hat{\psi}_1(1) \hat{\psi}_2(1) + q^2 \sum_{n=0}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n) \\ &\quad - q \left(\sum_{n=2}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n-2) + \sum_{n=0}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n+2) \right) + \sum_{n=2}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n) . \end{aligned}$$

Now

$$\hat{\psi}(n-2) + \hat{\psi}(n+2) = 2\hat{\psi}(n) - 4(\psi \sin^2)^\wedge(n) .$$

Therefore we have

$$\begin{aligned} &(q+1) \hat{\psi}_1(1) \hat{\psi}_2(1) + q^2 \sum_{n=1}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n) \\ &\quad - q \left(2 \sum_{n=1}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n) - 4 \sum_{n=1}^{\infty} \hat{\psi}_1(n) (\psi_2 \sin^2)^\wedge(n) \right) \\ &\quad - q (-\hat{\psi}_1(1) \hat{\psi}_2(-1) + \hat{\psi}_1(0) \hat{\psi}_2(2)) + \sum_{n=1}^{\infty} \hat{\psi}_1(n) \hat{\psi}_2(n) - \hat{\psi}_1(1) \hat{\psi}_2(1) . \end{aligned}$$

Recalling Parseval's formula

$$\sum_{n=1}^{\infty} \hat{\psi}_1(n) \overline{\hat{\psi}_2(n)} = \frac{1}{2\pi} \int_0^\pi \psi_1(\theta) \hat{\psi}_2(\theta) d\theta$$

we get

$$-\frac{1}{2\pi} \int_0^\pi \psi_1(\theta) \psi_2(\theta) ((q-1)^2 + 4q \sin^2(\theta)) d\theta,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. SPECTRAL DECOMPOSITION

Let E be the space of functions F_ψ with $\psi \in L^2([0, \pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T . Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E.$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $\langle g, F_\psi \rangle = 0$ for all ψ , i.e.,

$$\begin{aligned} 0 &= \frac{1}{q+1} g(0) \frac{1}{2\pi} \int_{-\pi}^\pi i(q+1) \sin \theta \psi(\theta) d\theta \\ &+ \sum_{n=1}^{\infty} g(n) \frac{1}{2\pi} \int_{-\pi}^\pi \psi(\theta) i(\sin((n+1)\theta) - q \sin((n-1)\theta)) d\theta q^{-\frac{n}{2}} \\ &= g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) (\hat{\psi}(n+1) - q \hat{\psi}(n-1)) q^{-\frac{n}{2}}. \end{aligned}$$

Therefore

$$g(0) \hat{\psi}(1) + \sum_{n=1}^{\infty} g(n) \hat{\psi}(n+1) q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}},$$

or (as $\hat{\psi}(0) = 0$)

$$\sum_{n=1}^{\infty} g(n-1) \hat{\psi}(n) q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1) \hat{\psi}(n) q^{-\frac{n-1}{2}}.$$

Since $\psi \in L^2([0, \pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over