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Autor: Efrat, Isaac
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Then, for $n \geq 1$,

$$\begin{aligned} \tilde{f}_\theta(n) &= q^{\frac{n}{2}} i((q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta)) \\ &= q^{\frac{n}{2}} i(\sin((n+1)\theta) - q\sin((n-1)\theta)) \,, \end{aligned} \tag{5}$$

and

$$\tilde{f}_\theta(0) = (q+1) i \sin\theta \ .$$

PROPOSITION 3.6. *The functions* $\tilde{f}_\theta, 0 < \theta < \pi$, *are not in* $L^2(F)$.

Proof. It is sufficient to show that

$$(q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta) \not\to 0 \quad \text{as} \quad n \to \infty \ .$$

This is the dot product of the two vectors

$$v_1 = ((q+1)\sin\theta, -(q-1)\cos\theta) \quad \text{and} \quad v_2 = (\cos(n\theta), \sin(n\theta)) \ .$$

Since v_2 is not a constant function of n, we see that cosine of the angle between v_1 and v_2 is bounded away from 0 for arbitrarily large n.

Combining Propositions 3.5 and 3.6 we conclude

COROLLARY 3.7. *The discrete spectrum of* T *consists of the numbers* $\pm(q+1)$, *whose corresponding eigenfunctions (given in Example 3.1) span two one-dimensional eigenspaces of* $L^2(F)$.

Unlike the typical f_λ with $|\lambda| > 2\sqrt{q}$, those with $|\lambda| < 2\sqrt{q}$ satisfy

$$f_\lambda = O(q^{\frac{n}{2}}) \ .$$

Our goal now is to show that these are approximate eigenfunctions that can be used to completely decompose $L^2(F)$.

4. CONTINUOUS SPECTRA

We wish to embed $L^2([0,\pi])$ with an appropriate measure into $L^2(F)$. To this end, let $\psi \in L^2([0,\pi])$ and $\tilde{f}_\theta(n)$ be extended as odd functions of $\theta \in [-\pi,\pi]$, and define

$$F_\psi(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\theta)\tilde{f}_\theta(n)\, d\theta \ .$$

THEOREM 4.1. *F_ψ is in $L^2(F)$ and we have the Plancherel formula*

$$< F_{\psi_1}, F_{\psi_2} > \; = \; < \psi_1, \psi_2 > \,, \tag{6}$$

where the inner product on the right is

$$\frac{1}{2\pi}\int_0^\pi \psi_1(\theta)\bar{\psi}_2(\theta)\,((q-1)^2 + 4q\sin^2\theta)\,d\theta \,.$$

Proof. We first note that

$$\frac{1}{2\pi}\int_\pi^\pi \psi(\theta)\,(i\sin((n+1)\theta)) - qi\sin((n-1)\theta))\,d\theta = q\hat{\psi}(n-1) - \hat{\psi}(n+1)\,,$$

where $\hat{\psi}(n)$ is the n-th Fourier coefficient of ψ. Therefore

$$< F_{\psi_1}, \bar{F}_{\psi_2} > \; = \frac{1}{q+1}\,(q+1)^2\hat{\psi}_1(1)\hat{\psi}_2(1)$$

$$+ \sum_{n=1}^{\infty} (q^2\hat{\psi}_1(n-1)\hat{\psi}_2(n-1) - q\hat{\psi}_1(n+1)\hat{\psi}_2(n-1)$$

$$- q\hat{\psi}_1(n-1)\hat{\psi}_2(n+1) + \hat{\psi}_1(n+1)\hat{\psi}_2(n+1))$$

$$= (q+1)\hat{\psi}_1(1)\hat{\psi}_2(1) + q^2\sum_{n=0}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n)$$

$$- q(\sum_{n=2}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n-2) + \sum_{n=0}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n+2)) \; + \; \sum_{n=2}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n)\,.$$

Now

$$\hat{\psi}(n-2) + \hat{\psi}(n+2) = 2\hat{\psi}(n) - 4(\psi\sin^2)^\wedge(n)\,.$$

Therefore we have

$$(q+1)\hat{\psi}_1(1)\hat{\psi}_2(1) + q^2\sum_{n=1}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n)$$

$$- q(2\sum_{n=1}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n) - 4\sum_{n=1}^{\infty}\hat{\psi}_1(n)(\psi_2\sin^2)^\wedge(n))$$

$$- q(-\hat{\psi}_1(1)\hat{\psi}_2(-1) + \hat{\psi}_1(0)\hat{\psi}_2(2)) + \sum_{n=1}^{\infty}\hat{\psi}_1(n)\hat{\psi}_2(n) \; - \; \hat{\psi}_1(1)\hat{\psi}_2(1)\,.$$

Recalling Parseval's formula

$$\sum_{n=1}^{\infty}\hat{\psi}_1(n)\overline{\hat{\psi}_2(n)} = \frac{1}{2\pi}\int_0^\pi \psi_1(\theta)\hat{\psi}_2(\theta)\,d\theta$$

we get

$$-\frac{1}{2\pi}\int_0^{\pi}\psi_1(\theta)\psi_2(\theta)\left((q-1)^2+4q\sin^2(\theta)\right)d\theta\ ,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. Spectral decomposition

Let E be the space of functions F_ψ with $\psi \in L^2([0,\pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T. Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E\ .$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $< g, F_\psi > = 0$ for all ψ, i.e.,

$$0 = \frac{1}{q+1}\,g(0)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} i(q+1)\sin\theta\psi(\theta)d\theta$$

$$+\sum_{n=1}^{\infty} g(n)\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(\theta)i(\sin((n+1)\theta)-q\sin((n-1)\theta))d\theta\, q^{-\frac{n}{2}}$$

$$= g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)(\hat{\psi}(n+1)-q\hat{\psi}(n-1))q^{-\frac{n}{2}}\ .$$

Therefore

$$g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)\hat{\psi}(n+1)q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ ,$$

or (as $\hat{\psi}(0)=0$)

$$\sum_{n=1}^{\infty} g(n-1)\hat{\psi}(n)q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ .$$

Since $\psi \in L^2([0,\pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over