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Artikel: AUTOMORPHIC SPECTRA ON THE TREE OF PGL_2
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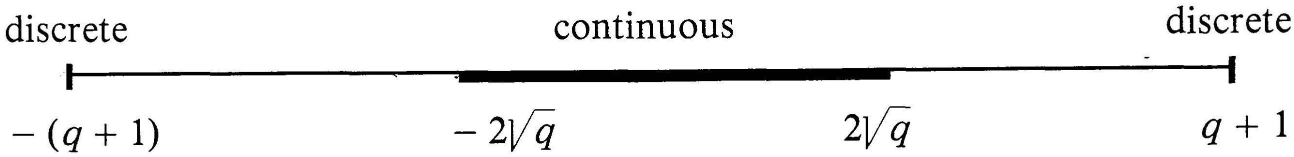
3. *The above describes a decomposition* $L^2(F) = R \oplus E$ *(Theorem 5.1), made explicit in Theorem 5.3 (compare* [L]*). In particular, the spectrum of* T *on* $L^2(F)$ *is*



1. The tree of $PGL_2(K)$

The material in this section is adapted from Serre [S] and Weil [W].

An O-lattice in K^2 is a set

$$L = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in O\}$$

with v_1, v_2 a basis for K^2. We can associate to L the matrix $(v_1, v_2) \in GL_2(K)$ and different choices of bases v_1, v_2 will give cosets in $GL_2(K)/GL_2(O)$. Two lattices L and L' are said to be equivalent if $L' = aL$ for some $a \in K^\times$. We thus have a natural correspondence between equivalence classes of O-lattices in K^2 and points in X.

We define a graph structure on X. Let Λ and Λ' be two equivalence classes of lattices. We say that Λ and Λ' are adjacent if there exist representatives $L \in \Lambda, L' \in \Lambda'$ such that

$$L' \subset L \quad \text{and} \quad L/L' \cong k\,. \tag{1}$$

THEOREM 1.1 ([S]). *The graph whose set of vertices is* X *and whose edges are the pairs* (Λ, Λ') *satisfying (1) is the (infinite)* $(q+1)$*-regular tree.*

We seek a more explicit realization of X. Let B be the Borel subgroup of G_K consisting of the matrices whose bottom row is $(0, 1)$. Then the Iwasawa decomposition is

$$G_K = BG_O\,,$$

but it is not difficult to see that in fact any coset in X has a representative of the form

$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$$

with $x \in K$ and a uniquely determined $n \in \mathbf{Z}$.

In these coordinates we can write down the $q + 1$ vertices of X that are adjacent to a typical vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$. They are

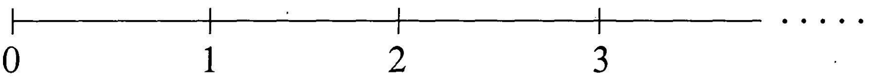
$$\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t^{n-1} & \xi t^n + x \\ 0 & 1 \end{pmatrix}, \quad \xi \in k .$$

The group G_K acts on the tree X as a group of automorphisms. We can therefore define a graph structure on the quotient F for the action of Γ on X.

THEOREM 1.2 ([S], [W]). *The quotient graph* $F = \Gamma \backslash X$ *is given by (the cosets of)*

$$\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \quad n \geqslant 0 ,$$

so that F *is the tree*



In fact, the vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$ corresponds to n, and so if $n \geqslant 1$, its neighbor $\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}$ corresponds to $n + 1$ while the other q neighbors are represented by $n - 1$. If $n = 0$, all neighbors correspond to 1.

2. THE OPERATOR T

Let μ be the Haar measure on G_K normalized so that $\mu(G_O) = q(q - 1)$. We compute the measure of F induced from μ. Since

$$F = \Gamma \backslash X = \Gamma \backslash G_K / G_O$$

we have

$$\Gamma \backslash G_K = \cup_{s \in F} s G_O ,$$

where

$$s G_O = \{ \Gamma s u \mid u \in G_O \} \subset \Gamma \backslash G_K .$$

The point measure at s will be the measure of sG_O in the quotient space $\Gamma \backslash G_K$. Now we have a correspondence