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AUTOMORPHIC SPECTRA ON THE TREE OF PGL_2

by Isaac EFRAT [1])

0. INTRODUCTION

Our aim in this paper is to give a complete development of the spectral theory of functions on the Bruhat-Tits building attached to PGL_2 of a function field over a finite field, which are automorphic with respect to the associated modular group. This set-up may be viewed as the simplest, nontrivial case to which the theory of automorphic forms on GL_2 ([JL]) applies. Using only elementary means, we derive an explicit description of the resulting theory, with emphasis on the underlying spectral decomposition and the distinction between discrete and continuous spectra. This approach has in turn been instrumental in our recent work on the existence problem of cusp forms and their deformation theory ([E1], [E2]).

To describe this set-up in some detail, let k be the finite field with q elements. The norm at infinity of the field of rational functions $k(t)$ is given by

$$| f/g | = q^{\deg(f) - \deg(g)} ,$$

where f and g are polynomials in $k[t]$. The completion with respect to this norm is the field K of Laurent series in t^{-1}

$$\sum_{n=-N}^{\infty} a_n t^{-n} , \qquad a_n \in k ,$$

and those for which $N \geqslant 0$ form the maximal compact subring O of the local integers in K.

Consider the group $G_K = PGL_2(K)$ of all 2×2 invertible matrices over K modulo the scalar matrices. The subgroup

$$G_O = PGL_2(O)$$

[1]) Supported by M.S.R.I. grant 8120790.

is a maximal compact subgroup, which gives rise to the metric space

$$X = G_K / G_O$$

on which G_K acts via isometries. Thus the subgroup Γ of all elements with polynomial entries is a discrete subgroup that acts on X, resulting in a quotient $F = \Gamma \backslash X$. By an automorphic function we mean a Γ-invariant function on X, which is therefore just a function on F.

There is a natural operator T on finitely supported functions on X, which generates the Hecke algebra of operators that commute with the isometries of X. This operator has been studied by Cartier ([C1], [C2]) with emphasis on its spherical and harmonic functions. Here we bring Γ into play and study the spectral theory of T as an operator on $L^2(F)$. Specifically, it is our aim to give an explicit basis of $L^2(F)$ consisting of automorphic functions that are eigenfunctions of T.

This set-up stands in precise analogy with the classical situation of the modular group $SL_2(\mathbf{Z})$ acting on the upper half plane H. The operator T is then analogous to the Laplace-Beltrami operator of H. It is known (see [I], [T] for expositions) that in this case we have a decomposition into invariant subspaces

$$L^2(F) = R \oplus C \oplus E\,.$$

Here $R \oplus C$ (resp. E) is the subspace spanned by discrete (resp. continuous) eigenfunctions. More precisely, R is simply the one-dimensional space of constant functions. C is the span of the non-constant discrete eigenfunctions, which here are all cusp forms, meaning that they decay rapidly at the cusp of $SL_2(\mathbf{Z}) \backslash H$. This space can be shown to be infinite dimensional. Lastly, E is generated by functions $E(z, 1/2 + it)$ where $E(z, s)$ with $z \in H$, $s \in \mathbf{C}$ is the Eisenstein series attached to $SL_2(\mathbf{Z})$.

Bearing this analogy in mind, our main results are:

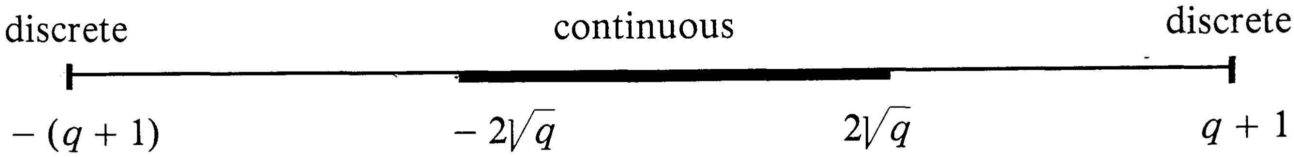
1. *The discrete eigenfunctions of* T *generate a two-dimensional subspace* R, *explicitly given by Proposition 3.5. In particular,* Γ *admits no cusp forms.* This is a special case of much more general dimension formulae (see [D], [HLW], [Sch]).

2. *The eigenfunctions in the continuous spectrum span a subspace* E *which is an isometric image of* $L^2([0, \pi])$ *with respect to the measure*

$$\frac{1}{2\pi}\left((q-1)^2 + 4q\sin^2\theta\right)d\theta$$

(Theorem 4.1).

3. *The above describes a decomposition* $L^2(F) = R \oplus E$ *(Theorem 5.1), made explicit in Theorem 5.3 (compare* [L]*). In particular, the spectrum of* T *on* $L^2(F)$ *is*



1. The tree of $PGL_2(K)$

The material in this section is adapted from Serre [S] and Weil [W].

An O-lattice in K^2 is a set

$$L = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in O\}$$

with v_1, v_2 a basis for K^2. We can associate to L the matrix $(v_1, v_2) \in GL_2(K)$ and different choices of bases v_1, v_2 will give cosets in $GL_2(K)/GL_2(O)$. Two lattices L and L' are said to be equivalent if $L' = aL$ for some $a \in K^\times$. We thus have a natural correspondence between equivalence classes of O-lattices in K^2 and points in X.

We define a graph structure on X. Let Λ and Λ' be two equivalence classes of lattices. We say that Λ and Λ' are adjacent if there exist representatives $L \in \Lambda, L' \in \Lambda'$ such that

$$L' \subset L \quad \text{and} \quad L/L' \cong k\,. \tag{1}$$

THEOREM 1.1 ([S]). *The graph whose set of vertices is* X *and whose edges are the pairs* (Λ, Λ') *satisfying (1) is the (infinite)* $(q+1)$*-regular tree.*

We seek a more explicit realization of X. Let B be the Borel subgroup of G_K consisting of the matrices whose bottom row is $(0, 1)$. Then the Iwasawa decomposition is

$$G_K = BG_O\,,$$

but it is not difficult to see that in fact any coset in X has a representative of the form

$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$$

with $x \in K$ and a uniquely determined $n \in \mathbf{Z}$.

In these coordinates we can write down the $q+1$ vertices of X that are adjacent to a typical vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$. They are

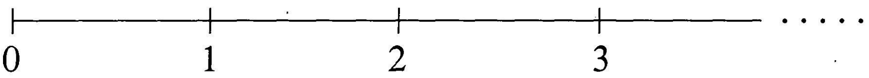
$$\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t^{n-1} & \xi t^n + x \\ 0 & 1 \end{pmatrix}, \quad \xi \in k .$$

The group G_K acts on the tree X as a group of automorphisms. We can therefore define a graph structure on the quotient F for the action of Γ on X.

THEOREM 1.2 ([S], [W]). *The quotient graph* $F = \Gamma \backslash X$ *is given by (the cosets of)*

$$\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \quad n \geqslant 0 ,$$

so that F *is the tree*



In fact, the vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$ corresponds to n, and so if $n \geqslant 1$, its neighbor $\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}$ corresponds to $n+1$ while the other q neighbors are represented by $n-1$. If $n = 0$, all neighbors correspond to 1.

2. THE OPERATOR T

Let μ be the Haar measure on G_K normalized so that $\mu(G_O) = q(q-1)$. We compute the measure of F induced from μ. Since

$$F = \Gamma \backslash X = \Gamma \backslash G_K / G_O$$

we have

$$\Gamma \backslash G_K = \cup_{s \in F} s G_O ,$$

where

$$s G_O = \{ \Gamma s u \mid u \in G_O \} \subset \Gamma \backslash G_K .$$

The point measure at s will be the measure of sG_O in the quotient space $\Gamma \backslash G_K$. Now we have a correspondence

$$sG_O \simeq s^{-1}\Gamma_s s \backslash G_O \,,$$

where $\Gamma_s = \Gamma \cap sG_O s^{-1}$ is the finite subgroup of Γ that stabilizes s. Thus

$$\mu(sG_O) = \frac{\mu(G_O)}{|\Gamma_s|} \,.$$

It is not hard to check that if $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $|\Gamma_s| = q(q^2-1)$, while for $s = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}$, $n \geqslant 1$, $|\Gamma_s| = (q-1)q^{n+1}$. We therefore put mass $\dfrac{1}{q+1}$ at the vertex 0 and q^{-n} at the vertices $n = 1, 2, \ldots$, so that if f and g are functions on $F = \{0, 1, 2, \ldots\}$ then

$$(2) \qquad < f, g > \; = \int_F f\bar{g}\, d\mu = \frac{1}{q+1} f(0)\bar{g}(0) + \sum_{n=1}^{\infty} f(n)\bar{g}(n) q^{-n} \,.$$

The algebra of operators on functions on the tree X that commute with the automorphisms of X is generated by the operator

$$(Tf)(s) = \sum_{s' \text{ is adjacent to } s} f(s')$$

(see [C2]). The operator $(q+1)I - T$ is the Laplacian on X.

If f is Γ-automorphic, and therefore can be thought of as a function on F, then T operates on f by

$$(3) \qquad (Tf)(n) = \begin{cases} qf(n-1) + f(n+1)\,, & \text{if } \; n \geqslant 1\,, \\ (q+1)f(1), & \text{if } \; n = 0\,. \end{cases}$$

PROPOSITION 2.1. *T is a self-adjoint operator on $L^2(F)$ with respect to the measure μ.*

Proof. If the series $\| f \|^2$ converges, then Cauchy's inequality implies that the four series in $\| Tf \|^2$ also converge. Thus T maps $L^2(F)$ into itself. Now

$$< Tf, \bar{g} > \; = \frac{1}{q+1}(q+1)f(1)g(0)$$

$$+ \sum_{n=1}^{\infty} (qf(n-1)g(n) + f(n+1)g(n))q^{-n}$$

$$= f(1)g(0) + \sum_{n=0}^{\infty} qf(n)g(n+1)q^{-(n+1)} + \sum_{n=2}^{\infty} f(n)g(n-1)q^{-(n-1)}$$

$$= f(1)g(0) + f(0)g(1) + \sum_{n=1}^{\infty} f(n)g(n+1)q^{-n}$$

$$+ q\sum_{n=1}^{\infty} f(n)g(n-1)q^{-n} - f(1)g(0)$$

$$= f(0)g(1) + \sum_{n=0}^{\infty} (f(n)qg(n-1) + f(n)g(n+1))q^{-n} = \langle f, T\bar{g}\rangle .$$

3. Eigenfunctions

An automorphic eigenfunction of T on X with eigenvalue λ is a function on F that satisfies

$$\lambda f(0) = (q+1)f(1) ,$$

$$\lambda f(n) = qf(n-1) + f(n+1) , \quad n \geqslant 1 .$$

If we write $u(n) = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$ and normalize $u(0) = \begin{pmatrix} \lambda \\ q+1 \end{pmatrix}$, we obtain the recursion

$$u(n) = A^n u(0)$$

with

$$A = \begin{pmatrix} \lambda & -q \\ 1 & 0 \end{pmatrix} .$$

Let $x_1, x_2 = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4q})$ be the characteristic roots of A and assume that $x_1 \neq x_2$, i.e., that $\lambda \neq \pm 2\sqrt{q}$. Solving the recursion we get

PROPOSITION 3.1. *The eigenfunctions on F with eigenvalue λ are the multiples of the function*

$$(4) \quad f_\lambda(n) = \begin{cases} \dfrac{1}{x_1 - x_2} (\lambda(x_1^n - x_2^n) - q(q+1)(x_1^{n-1} - x_2^{n-1})) , & \text{if } n \geqslant 1 \\ q+1 \quad \text{if} \quad n = 0 . \end{cases}$$

Example 3.2. If $\lambda = q+1$ then $x_1 = q, x_2 = 1$ and

$$f_{q+1}(n) \equiv q+1 \ ,$$

generating the space of constant functions. If $\lambda = -(q+1)$, then

$$f_{-(q+1)}(n) = (-1)^n(q+1) \ .$$

Example 3.3. When $\lambda = 2\sqrt{q}$ we can solve directly to get

$$f_{2\sqrt{q}}(n) = (q+1-(q-1)n)q^{\frac{n}{2}},$$

and similarly

$$f_{-2\sqrt{q}}(n) = (-1)^n(q+1-(q-1)n)q^{\frac{n}{2}} \ .$$

Remark 3.4. Since our tree is bipartite, we expect f_λ to be related to $f_{-\lambda}$ by a factor of $(-1)^n$ (compare [B, § 8]). This can be seen from (4).

PROPOSITION 3.5. *The only eigenvalues λ with $|\lambda| > 2\sqrt{q}$ for which f_λ is in $L^2(F)$ are $\lambda = \pm(q+1)$.*

Proof. Recalling (2) we see that if $f_\lambda \in L^2(F)$ then

$$f_\lambda(n) = o(q^{\frac{n}{2}}) \quad \text{as} \quad n \to \infty \ .$$

Now

$$(x_1 - x_2)f_\lambda(n) = x_1^{n-1}(\lambda x_1 - q(q+1)) - x_2^{n-1}(\lambda x_2 - q(q+1)) \ .$$

Assuming with no loss of generality that $|x_1| > \sqrt{q}$, $|x_2| = \dfrac{q}{|x_1|} < \sqrt{q}$, then $x_2^{n-1}(\lambda x_2 - q(q+1)) = o(q^{\frac{n}{2}})$, so that we must have

$$\lambda x_1 - q(q+1) = 0 \ ,$$

i.e., $\lambda = \pm(q+1)$. Conversely, f_{q+1} and $f_{-(q+1)}$ are clearly in $L^2(F)$.

We turn our attention to λ with $|\lambda| < 2\sqrt{q}$. Then $x_2 = \bar{x}_1$, $|x_1| = \sqrt{q}$ and we let $x_1 = \sqrt{q}e^{i\theta}$. Then $\lambda = 2\sqrt{q}\cos\theta, 0 < \theta < \pi$. We renormalize and define

$$\tilde{f}_\theta = \frac{x_1 - x_2}{2\sqrt{q}} f_{2\sqrt{q}\cos\theta} \ .$$

Then, for $n \geqslant 1$,

$$
\begin{aligned}
\tilde{f}_\theta(n) &= q^{\frac{n}{2}} i((q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta)) \\
&= q^{\frac{n}{2}} i(\sin((n+1)\theta) - q\sin((n-1)\theta)) ,
\end{aligned} \tag{5}
$$

and

$$\tilde{f}_\theta(0) = (q+1) i \sin\theta .$$

PROPOSITION 3.6. *The functions* $\tilde{f}_\theta, 0 < \theta < \pi$, *are not in* $L^2(F)$.

Proof. It is sufficient to show that

$$(q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta) \not\to 0 \quad \text{as} \quad n \to \infty .$$

This is the dot product of the two vectors

$$v_1 = ((q+1)\sin\theta, -(q-1)\cos\theta) \quad \text{and} \quad v_2 = (\cos(n\theta), \sin(n\theta)) .$$

Since v_2 is not a constant function of n, we see that cosine of the angle between v_1 and v_2 is bounded away from 0 for arbitrarily large n.

Combining Propositions 3.5 and 3.6 we conclude

COROLLARY 3.7. *The discrete spectrum of* T *consists of the numbers* $\pm(q+1)$, *whose corresponding eigenfunctions (given in Example 3.1) span two one-dimensional eigenspaces of* $L^2(F)$.

Unlike the typical f_λ with $|\lambda| > 2\sqrt{q}$, those with $|\lambda| < 2\sqrt{q}$ satisfy

$$f_\lambda = O(q^{\frac{n}{2}}) .$$

Our goal now is to show that these are approximate eigenfunctions that can be used to completely decompose $L^2(F)$.

4. CONTINUOUS SPECTRA

We wish to embed $L^2([0,\pi])$ with an appropriate measure into $L^2(F)$. To this end, let $\psi \in L^2([0,\pi])$ and $\tilde{f}_\theta(n)$ be extended as odd functions of $\theta \in [-\pi, \pi]$, and define

$$F_\psi(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\theta) \tilde{f}_\theta(n) d\theta .$$

THEOREM 4.1. *F_ψ is in $L^2(F)$ and we have the Plancherel formula*

$$<F_{\psi_1}, F_{\psi_2}> \; = \; <\psi_1, \psi_2> \; , \tag{6}$$

where the inner product on the right is

$$\frac{1}{2\pi} \int_0^\pi \psi_1(\theta)\bar{\psi}_2(\theta)((q-1)^2 + 4q\sin^2\theta)d\theta \; .$$

Proof. We first note that

$$\frac{1}{2\pi} \int_\pi^\pi \psi(\theta)(i\sin((n+1)\theta)) - qi\sin((n-1)\theta))d\theta = q\hat{\psi}(n-1) - \hat{\psi}(n+1) \; ,$$

where $\hat{\psi}(n)$ is the n-th Fourier coefficient of ψ. Therefore

$$<F_{\psi_1}, \bar{F}_{\psi_2}> \; = \frac{1}{q+1}(q+1)^2\hat{\psi}_1(1)\hat{\psi}_2(1)$$

$$+ \sum_{n=1}^{\infty} (q^2\hat{\psi}_1(n-1)\hat{\psi}_2(n-1) - q\hat{\psi}_1(n+1)\hat{\psi}_2(n-1)$$

$$- q\hat{\psi}_1(n-1)\hat{\psi}_2(n+1) + \hat{\psi}_1(n+1)\hat{\psi}_2(n+1))$$

$$= (q+1)\hat{\psi}_1(1)\hat{\psi}_2(1) + q^2 \sum_{n=0}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n)$$

$$- q(\sum_{n=2}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n-2) + \sum_{n=0}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n+2)) \; + \; \sum_{n=2}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n) \; .$$

Now

$$\hat{\psi}(n-2) + \hat{\psi}(n+2) = 2\hat{\psi}(n) - 4(\psi\sin^2)^\wedge(n) \; .$$

Therefore we have

$$(q+1)\hat{\psi}_1(1)\hat{\psi}_2(1) + q^2 \sum_{n=1}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n)$$

$$- q(2\sum_{n=1}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n) - 4\sum_{n=1}^{\infty} \hat{\psi}_1(n)(\psi_2\sin^2)^\wedge(n))$$

$$- q(-\hat{\psi}_1(1)\hat{\psi}_2(-1) + \hat{\psi}_1(0)\hat{\psi}_2(2)) + \sum_{n=1}^{\infty} \hat{\psi}_1(n)\hat{\psi}_2(n) \; - \; \hat{\psi}_1(1)\hat{\psi}_2(1) \; .$$

Recalling Parseval's formula

$$\sum_{n=1}^{\infty} \hat{\psi}_1(n)\overline{\hat{\psi}_2(n)} = \frac{1}{2\pi} \int_0^\pi \psi_1(\theta)\hat{\psi}_2(\theta)d\theta$$

we get

$$-\frac{1}{2\pi}\int_0^{\pi}\psi_1(\theta)\psi_2(\theta)\left((q-1)^2+4q\sin^2(\theta)\right)d\theta\ ,$$

and since $\bar{F}_{\psi_2} = -F_{\bar{\psi}_2}$ we obtain (6).

5. Spectral decomposition

Let E be the space of functions F_ψ with $\psi \in L^2([0,\pi])$. It follows from §4 that E is a subspace of $L^2(F)$, invariant with respect to T. Further, let R be the two dimensional subspace generated by the discrete spectrum according to Corollary 3.7.

THEOREM 5.1. *We have a direct sum decomposition into invariant subspaces*

$$L^2(F) = R \oplus E\ .$$

Proof. The two spaces are easily seen to be orthogonal. We show that $E^\perp = R$. Let $g \in L^2(F)$ such that $< g, F_\psi > = 0$ for all ψ, i.e.,

$$0 = \frac{1}{q+1}\,g(0)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} i(q+1)\sin\theta\psi(\theta)d\theta$$

$$+\sum_{n=1}^{\infty} g(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(\theta)i(\sin((n+1)\theta)-q\sin((n-1)\theta))d\theta\, q^{-\frac{n}{2}}$$

$$= g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)(\hat{\psi}(n+1)-q\hat{\psi}(n-1))q^{-\frac{n}{2}}\ .$$

Therefore

$$g(0)\hat{\psi}(1)+\sum_{n=1}^{\infty} g(n)\hat{\psi}(n+1)q^{-\frac{n}{2}} = \sum_{n=0}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ ,$$

or (as $\hat{\psi}(0) = 0$)

$$\sum_{n=1}^{\infty} g(n-1)\hat{\psi}(n)q^{-\frac{n-1}{2}} = \sum_{n=1}^{\infty} g(n+1)\hat{\psi}(n)q^{-\frac{n-1}{2}}\ .$$

Since $\psi \in L^2([0,\pi])$ and $g \in L^2(F)$, this can be viewed as an equality of inner products in the space l^2 of square integrable sequences. Now as ψ varies over

$L^2([0,\pi])$, the sequences $\{\hat{\psi}(n)\}$ vary over l^2. Since the latter is a Hilbert space, it follows that for all n,

$$g(n-1) = g(n+1) .$$

Let $a = \frac{1}{2}(g(0)+g(1))$, $b = \frac{1}{2}(g(0)-g(1))$. Then

$$g(n) = a + (-1)^n b ,$$

a typical member of R.

COROLLARY 5.2. *The spectrum of* T *on* $L^2(F)$ *is the subset of* $\mathbf{R}$ *described in the Introduction.*

We wish to make this decomposition explicit, that is, given $g \in E$ to find the ψ such that $g = F_\psi$ (compare [H]). Let $\phi(\theta)$ be the characteristic function of $[\theta_0, \theta_0 + h] \subset [0,\pi]$. Then

$$F_\phi(n) = \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n)\, d\theta .$$

By the Plancherel formula (6),

$$< g, F_\phi > = < g, \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \tilde{f}_\theta(n)\, d\theta >$$

$$= \frac{1}{2\pi} \int_0^\pi \psi(\theta)\phi(\theta)\left((q-1)^2 + 4q\sin^2\theta\right) d\theta$$

$$= \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+h} \psi(\theta)\left((q-1)^2 + 4q\sin^2\theta\right) d\theta .$$

Supposing further that g belongs in the dense subspace $E \cap L^1(F)$, we divide by h and let $h \to 0$ to obtain

$$< g, \tilde{f}_{\theta_0} > = \psi(\theta_0)\left((q-1)^2 + 4q\sin^2\theta\right)/2\pi .$$

Hence:

THEOREM 5.3. *Let* u_1 *and* u_2 *be the orthonormal basis of* R *given by*

$$u_1(n) \equiv \sqrt{\frac{q^2-1}{2q}}, \quad u_2(n) = \sqrt{\frac{q^2-1}{2q}}(-1)^n .$$

Then the spectral resolution of $L^2(F)$ *reads*

(7)

$$g(n) = \sum_{i=1,2} < g, u_i > u_i(n) + 2\pi \int_0^{\pi} < g, \tilde{f}_\theta > \tilde{f}_\theta(n) \frac{d\theta}{(q-1)^2 + 4q \sin^2 \theta} .$$

We end this paper by showing that, as one might expect from the theory of Eisenstein series, the eigenfunctions f_λ can be parametrized as a family of functions that depend holomorphically on a complex parameter. Precisely, let

$$E(n, s) = q^{ns}(q^{s-1} - q^{1-s}) + q^{n(1-s)}(q^s - q^{-s}) .$$

Then $E(n, s)$ is entire in s and satisfies the functional equation

$$E(n, s) = - E(n, 1 - s) .$$

Furthermore, a direct computation shows that

$$(TE)(n, s) = (q^s + q^{1-s})E(n, s) .$$

There are two ways in which $\lambda = q^s + q^{1-s}$ can be real. Write $s = \sigma + it$. If $t = \dfrac{k\pi}{\log q}$ then $\lambda = (-1)^k(q^\sigma + q^{1-\sigma})$, and in particular

$$\lambda = q + 1 (\lambda = -(q + 1))$$

if $\sigma = 1$ and k is even (k is odd). Otherwise we must have $\sigma = \dfrac{1}{2}$. If we write $t = \dfrac{\theta}{\log q}$ we obtain our $\lambda = 2\sqrt{q} \cos \theta$.

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