4. Diffeomorphism groups of some algebraic surfaces

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$$f^*\Phi_{c,\alpha,P}(X) = \Phi_{-c,-\alpha,P}(X) .$$

Applying §2(a) and (b) with $a = -c$ we get

$$f^*\Phi_{c,\alpha,P}(X) = -\Phi_{-c,\alpha,P}(X) = -(-1)^{c^2}\Phi_{c,\alpha,P}(X) .$$

By assumption $\Phi_{c,\alpha,P}(X) \not\equiv 0$, so we must have

$$(-1)^{c^2+1} = (-1)^d .$$

Now $d = 4c_2 - c^2 - 3(1 + p_g(X))$ implies that $(-1)^{p_g(X)} = 1$, i.e. $p_g(X) \equiv 0 \pmod 2$. This proves Theorem 6.

4. Diffeomorphism groups of some algebraic surfaces

In §1 we saw that the image of ψ contains the group $O_k'(L) \cdot \{\sigma_*, \mathrm{id}\}$ in many cases. In §2 we showed that under certain conditions $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$. Finally we proved in the previous section that for algebraic surfaces of odd geometric genus -1 is not induced by an orientation preserving diffeomorphism. It turns out that these facts suffice to determine the image of ψ.

Proposition 7. *Let X be a simply connected algebraic surface which satisfies the following conditions:*

(i) $O_k'(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X))$,

(ii) $\{\pm k_X\}$ *is invariant under* $\psi(\mathrm{Diff}_+(X))$,

(iii) $-1 \notin \psi(\mathrm{Diff}_+(X))$.

Then

$$\psi(\mathrm{Diff}_+(X)) = O_k'(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

Proof. Let $g = -\sigma_*$. Then $g \in O_k(L)$, but $g \notin \psi(\mathrm{Diff}_+(X))$, since $-1 \notin \psi(\mathrm{Diff}_+(X))$. Hence by (i), $g \notin O_k'(L)$. Therefore

$$O_k(L) = O_k'(L) \cdot \{g, \mathrm{id}\} .$$

Now let $h \in \psi(\mathrm{Diff}_+(X))$. By (ii) either $h(k) = k$ or $h(k) = -k$. In the first case $h \in O_k(L)$. Moreover, $h \in O_k'(L)$ since otherwise $h = gh_0$ for some $h_0 \in O_k'(L)$ which would imply $g \in \psi(\mathrm{Diff}_+(X))$, a contradiction. In the second case we have $h' = h\sigma_* \in O_k(L)$. By the same argument as before we see that $h' \in O_k'(L)$. Hence $h = h'\sigma_* \in O_k'(L) \cdot \{\sigma_*, \mathrm{id}\}$. This proves Proposition 7.

Putting everything together we get the main result of our paper.

THEOREM 8. *Let* X *be a simply connected algebraic surface with* $p_g(X) \equiv 1 \pmod 2$ *and* $k_X^2 \equiv 1 \pmod 2$. *If* X *is either*

(i) *a complete intersection, or*

(ii) *a Moishezon or Salvetti surface,*

then

$$\psi(\mathrm{Diff}_+(X)) = O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

Example. Consider a complete intersection surface X of multidegree $(d_1, \ldots, d_r)$. Using the formulas of [E4], we can translate the conditions of Theorem 8 into numerical conditions on the degrees d_i. The condition $k_X^2 \equiv 1 \pmod 2$ is equivalent to

(1) $$d_i \equiv 1 \pmod 2 \quad \text{for} \quad i = 1, \ldots, r .$$

Write $d_i = 2e_i + 1$ $(i = 1, \ldots, r)$. Then $p_g(X) \equiv 1 \pmod 2$ is equivalent to the following two conditions

(2) $$\sum_{i<j} e_i e_j \equiv 1 \pmod 2 ,$$

(3) $$\text{either } 3 \mid d_j \text{ for some } j,\ 1 \leqslant j \leqslant r, \quad \text{or} \quad 3 \mid \textstyle\sum_{i=1}^{r} e_i(e_i + 1) .$$

In particular there is an infinite sequence of complete intersection surfaces satisfying the conditions of Theorem 8, e.g. the surfaces with $(d_1, d_2) = (3, 3 + 4m)$, $m \in \mathbf{Z}$, $m \geqslant 0$.

We leave it to the reader to formulate similar conditions for the case (ii) of Theorem 8.

We shall give two further applications of the results of the first three sections.

Let X be a surface as in Theorem 1 and denote the symmetric bilinear form corresponding to q_X by $\langle\ ,\ \rangle$. Define $L' := \ker k_X = k_X^{\perp} \subset L$, and let $\Delta \subset L'$ be the set of vanishing cycles of X (cf. [EO]). The pair (L', Δ) is then a vanishing lattice in the sense of [E1, Definition (2.1)]. This means the following. If $\delta \in \Delta$, then $\langle \delta, \delta \rangle = -2$ and one has an associated reflection s_δ defined by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta$$

for all $x \in L'$. Let Γ_Δ denote the subgroup of $O(L')$ generated by these reflections s_δ, $\delta \in \Delta$. Then (L', Δ) satisfies the following conditions:

(i) $\langle \delta, \delta \rangle = -2$ for all $\delta \in \Delta$.

(ii) Δ generates L'.

(iii) Δ is a Γ_Δ-orbit.

(iv) Unless rank $L' = 1$, there exist $\delta_1, \delta_2 \in \Delta$ with $\langle \delta_1, \delta_2 \rangle = 1$.

As in Wall's paper [W] we can derive from a statement about $\psi(\mathrm{Diff}_+(X))$ a statement about the possibility of representing homology classes by embedded 2-spheres.

THEOREM 9. *Let X be an algebraic surface, and let $x \in H_2(X, \mathbf{Z})$ be a class with $q_X(x) = -2$. If x is represented by a differentiably embedded 2-sphere, then $\bar{x} \in \bar{k}_X^\perp$. Conversely, if X is a surface as in Theorem 1, if $x \in k_X^\perp$, and if there exists a class $y \in k_X^\perp$ with $\langle x, y \rangle = 1$, then x can be represented by a differentiably embedded 2-sphere.*

Proof. Let $x \in H_2(X, \mathbf{Z})$ be a class with $q_X(x) = -2$. Suppose that x is represented by a differentiably embedded 2-sphere S. Let $j: S \to X$ be the embedding. The normal bundle N_S of S in X can be regarded as a $U(1)$-bundle. Therefore the first Chern class $c_1(N_S)$ of the normal bundle is defined. Let $\xi \in H^2(X, \mathbf{Z})$ be the Poincaré dual of x. Then by [H, Theorem 4.8.1] $c_1(N_S) = j^*\xi$. If T_X and T_S denote the tangent bundles of X and S respectively, then we have $j^*\bar{k}_X = w_2(j^*T_X) = w_2(T_S) + w_2(N_S)$. This implies $\langle \bar{k}_X, \bar{\xi} \rangle = \overline{\chi(S) + \xi^2} = 0$ where $\chi(S)$ denotes the Euler characteristic of S. It follows that $\bar{x} \in \bar{k}_X^\perp$.

Conversely, let X be a surface as in Theorem 1. Then (L', Δ) is a complete vanishing lattice in the sense of [E1, Definition (2.2)]. This follows for complete intersection surfaces from [B, E3], for Moishezon surfaces from [M], and for Salvetti surfaces from [S] (see also [EO]). By [E1, Proposition (2.5)] we conclude that

$$\Delta = \{ v \in L' \mid q_X(v) = -2 \quad \text{and} \quad \langle v, L' \rangle = \mathbf{Z} \} .$$

Therefore, if $x \in L'$ and if there exists a $y \in L'$ with $\langle x, y \rangle = 1$, then $x \in \Delta$, i.e. x is a vanishing cycle. But vanishing cycles are certainly represented by spheres. This proves Theorem 9.

Remark. We have even proved more, namely that every x satisfying the latter conditions of Theorem 9 is a vanishing cycle.

Our second application concerns a question which was posed by E. Brieskorn. First we show:

PROPOSITION 10. *Let* X *be an algebraic surface as in Theorem 1. Let* $x \in \mathrm{Hom}(L, \mathbf{Z})$. *If* $\{\pm x\}$ *is invariant under* $\psi(\mathrm{Diff}_+(X))$, *then* $x = \lambda k_X$ *for some* $\lambda \in \mathbf{Q}$, *unless* $k_X^2 = 0, k_X \neq 0$.

Proof. Let $U \subset L'_{\mathbf{Q}} = L' \otimes \mathbf{Q}$ be a subspace of $L'_{\mathbf{Q}}$ which is invariant under $\psi(\mathrm{Diff}_+(X))$. We show that either $U = L'_{\mathbf{Q}}$ or U is contained in $(L'_{\mathbf{Q}})^{\perp} = \{v \in L'_{\mathbf{Q}} \mid \langle v, w \rangle = 0$ for all $w \in L'_{\mathbf{Q}}\}$. Let $\Delta \subset L'$ be the set of vanishing cycles. If $\delta \in \Delta$ is not orthogonal to U then there exists a $y \in U$ with $\langle \delta, y \rangle \neq 0$. From $s_\delta(y) = y + \langle y, \delta \rangle \delta \in U$ it follows that $\delta \in U$. Since Γ_Δ acts transitively on Δ, we obtain $\Delta \subset U$. But Δ generates L' so that we must have $L'_{\mathbf{Q}} = U$.

Now let $x \in \mathrm{Hom}(L, \mathbf{Z})$ be invariant up to sign under $\Gamma_\Delta \subset \psi(\mathrm{Diff}_+(X))$. If $k_X = 0$ then $L = L'$ and $(L')^{\perp} = \{0\}$. Hence it follows from what we have just shown that $x = 0$. If $k_X^2 \neq 0$ then we can write $x = \lambda k_X + x_0$ where $\lambda \in \mathbf{Q}$ and $x_0 \in \mathrm{Hom}(L', \mathbf{Z}) \otimes \mathbf{Q}$. Now k_X is invariant under Γ_Δ, so we see that $\{\pm x_0\}$ is invariant under Γ_Δ. Since $(L')^{\perp} = \{0\}$, it follows that $x_0 = 0$. This proves Proposition 10.

Now $(\overline{L'}, \bar{\Delta})$, where $\bar{\Delta} = \{\bar{\delta} \mid \delta \in \Delta\}$, is also a vanishing lattice. So we can derive the following proposition by the same arguments.

PROPOSITION 11. *Let* X *be an algebraic surface as in Theorem 1. Write* $k_X = n_X \kappa_X$ *for some primitive element* $\kappa_X \in \mathrm{Hom}(L, \mathbf{Z})$ *and some non-negative integer* n_X. *If* $x \in \mathrm{Hom}(L, \mathbf{Z})$ *is an element with* $\overline{g(x)} = \bar{x}$ *for all* $g \in \psi(\mathrm{Diff}_+(X))$, *then* $\bar{x} \in \{0, \overline{\kappa_X}\}$, *unless* $\kappa_X^2 = 0, \kappa_X \neq 0$.

Consider two simply connected algebraic surfaces X and X' with corresponding lattices L_X and $L_{X'}$. Let $h: X \to X'$ be an orientation preserving homeomorphism between X and X'. Then $h_*: L_X \to L_{X'}$ is an isometry. For a subgroup $G \subset O(L_{X'})$ we define $G^{h_*} := h_*^{-1} G h_*$. Note that we have $k_X^2 = k_{X'}^2$, since k_X^2 can be expressed in terms of the rank and signature of L_X.

THEOREM 12. *Let* X, X' *be algebraic surfaces as in Theorem 8, and suppose that* $k_X^2 = k_{X'}^2 \neq 0$. *Let* $h: X \to X'$ *be an orientation preserving homeomorphism. Then* $\psi(\mathrm{Diff}_+(X))$ *and* $\psi(\mathrm{Diff}_+(X'))^{h_*}$ *are conjugate subgroups in* $O(L_X)$ *if and only if the divisibilities of* k_X *and* $k_{X'}$ *in integral cohomology are equal.*

Proof. Suppose that $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+(X'))^{h_*}$ are conjugate by $g \in O(L_X)$, i.e. $\psi(\mathrm{Diff}_+(X)) = (\psi(\mathrm{Diff}_+(X'))^{h_*})^g$. Let $f \in \psi(\mathrm{Diff}_+(X'))$. Then

$$(h_*g)^{-1}f(h_*g)(k_X) = \pm k_X ,$$

since $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$. Therefore

$$f((h_*g)(k_X)) = \pm (h_*g)(k_X) .$$

This holds for every $f \in \psi(\mathrm{Diff}_+(X'))$, so that $\{\pm (h_*g)(k_X)\}$ is invariant under $\psi(\mathrm{Diff}_+(X'))$. We conclude from Proposition 10 that $(h_*g)(k_X) = \lambda k_{X'}$ for some $\lambda \in \mathbf{Q}$. Now

$$k_{X'}^2 = k_X^2 = ((h_*g)(k_X))^2 = \lambda^2 k_{X'}^2 ,$$

hence $\lambda = \pm 1$, and it follows that the divisibilities of k_X and $k_{X'}$ in integral cohomology are equal.

Conversely, let $k_X = n_X \kappa_X$ and $k_{X'} = n_{X'} \kappa_{X'}$ for some primitive elements $\kappa_X \in H^2(X, \mathbf{Z})$ and $\kappa_{X'} \in H^2(X', \mathbf{Z})$ and some nonnegative integers n_X and $n_{X'}$ respectively, and assume that n_X and $n_{X'}$ are equal. Let λ_X and $\lambda_{X'}$ be the Poincaré duals of κ_X and $\kappa_{X'}$, and let K and K' be the one-dimensional sublattices of L_X spanned by λ_X and $h_*^{-1}(\lambda_{X'})$ respectively. Let $g: K \to K'$ be the homomorphism defined by $g(\lambda_X) = h_*^{-1}(\lambda_{X'})$. Then $\langle g(\lambda_X), g(\lambda_X)\rangle = \langle \lambda_X, \lambda_X \rangle$, hence g is an isometry. Now $b_2^+(X) \geqslant 3$, so that by a generalization of Witt's theorem [N, § 1.14, in particular 1.14.4, § 1.16] g can be extended to an isometry $g \in O(L_X)$. One can easily verify that $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+ X'))^{h_*}$ are conjugate by g. This proves Theorem 12.

COROLLARY 13. *Let X be an algebraic surface as in Theorem 8. If an element $h \in O'(L)$ normalizes $\psi(\mathrm{Diff}_+(X))$, then h is contained in $\psi(\mathrm{Diff}_+(X))$.*

Proof. This follows from Theorem 12, because

$$\psi(\mathrm{Diff}_+(X)) = O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

Remark. Since -1 is not contained in $\psi(\mathrm{Diff}_+(X))$ but in the normalizer $\mathrm{Norm}(\psi(\mathrm{Diff}_+(X)))$ of $\psi(\mathrm{Diff}_+(X))$ we obtain from Corollary 13:

$$\mathrm{Norm}(\psi(\mathrm{Diff}_+(X)))/\psi(\mathrm{Diff}_+(X)) \cong \mathbf{Z}/2 \cong \{\pm \mathrm{id}\} .$$

Example. Let X be a complete intersection in $\mathbf{P}^6$ of multidegree $(7, 7, 5, 3)$. Let X' be the Salvetti surface $Y_4(10, 10, 6, 5; 5, 5, 3, 5)$. Both surfaces have $k^2 = 165375$, $p_g = 24499$, but the divisibilities of the canonical classes are 15 and 21 respectively. Therefore the surfaces are homeomorphic, but not diffeomorphic. Both surfaces satisfy the assumptions of Theorem 8.

We conclude from Theorem 12 that the subgroups of $O(L)$ corresponding to $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+(X'))$ are not conjugate. This example was found by a computer search.

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