2. Invariance of the canonical class

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **37 (1991)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let* X *be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X)) .$$

2. INVARIANCE OF THE CANONICAL CLASS

S.K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces X with $p_g(X) > 0$. We assume from now on that X is such a surface. There are two types of invariants according to the gauge group being $SU(2)$ or $SO(3)$.

Let us first recall the $SU(2)$-case. Principal $SU(2)$-bundles over X are classified by their second Chern class $c_2(P)$. For each $l > l_0$, using such a bundle with $c_2(P) = l$, Donaldson defines a polynomial

$$\Phi_l(X)\colon \mathrm{Sym}^d(L) \to \mathbf{Z}$$

of degree $d = d(l) = 4l - 3(p_g(X) + 1)$, which depends only on the underlying C^∞-structure of X and is invariant up to sign under $\psi(\mathrm{Diff}_+(X))$. Donaldson shows that these invariants are nontrivial for all sufficiently large l [D].

We will need the slightly more complicated $SO(3)$-invariants. The simple Lie group $SO(3)$ is isomorphic to $PU(2)$, so that one has an exact sequence

$$1 \to S^1 \to U(2) \to SO(3) \to 1 .$$

Let P be a principal $SO(3)$-bundle over X. Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbf{Z}/2)$ and the first Pontryagin class $p_1(P) \in H^4(X, \mathbf{Z})$.

Suppose that $w_2(P)$ is nonzero and choose an integral lifting c of $w_2(P)$, i.e. $c \in H^2(X, \mathbf{Z})$, $\bar{c} = w_2(P)$ (here $\bar{c}$ means the reduction of c modulo 2). Such a lifting exists since X is simply connected, and determines a $U(2)$-lifting $\hat{P}$ of P, i.e. a $U(2)$-bundle $\hat{P}$ with $\hat{P}/S^1 = P$ and with $c = c_1(\hat{P})$ [HH]. The Chern classes of $\hat{P}$ are related to the characteristic classes of P by $w_2(P) = \bar{c}_1(\hat{P})$ and $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$. In addition to this choose an element $\alpha \in \Omega$. Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c,\alpha,P}(X)\colon \mathrm{Sym}^d(L) \to \mathbf{Z}$$

of degree $d = -p_1(P) - 3(p_g(X)+1) = 4c_2(\hat{P}) - c^2 - 3(p_g(X)+1)$ with the following properties ([D], see also [OV]):

(a) $\Phi_{c,-\alpha,P}(X) = -\Phi_{c,\alpha,P}(X)$ where $-\alpha$ is the subspace corresponding to α with the opposite orientation.

(b) $\Phi_{c+2a,\alpha,P}(X) = \varepsilon(a)\Phi_{c,\alpha,P}(X)$ where

$$\varepsilon(a) = \begin{cases} 1 & \text{if} \quad \bar{a}^2 = 0 , \\ -1 & \text{if} \quad \bar{a}^2 \neq 0 . \end{cases}$$

(c) If $f: X' \to X$ is an orientation preserving diffeomorphism then

$$\Phi_{f^*(c), f^*(\alpha), f^*(P)}(X') = f^*\Phi_{c,\alpha,P}(X) .$$

Donaldson's nontriviality result for the $SU(2)$-invariants has been extended to the $SO(3)$-case by Zuo [Z]:

THEOREM 3 (Zuo). *Let X be a simply connected algebraic surface with $p_g(X) > 0$. If $c \in H^{1,1}(X, \mathbf{Z}), \bar{c} \neq 0$, and P is a principal $SO(3)$-bundle corresponding to a $U(2)$-bundle $\hat{P}$ with $c_1(\hat{P}) = c$ and $c_2(\hat{P})$ sufficiently large, then the polynomial $\Phi_{c,\alpha,P}(X)$ is nontrivial.*

Now suppose that X has a big monodromy group in the sense of Friedman and Morgan [FMM]. Then the $SU(2)$-invariants $\Phi_l(X)$ of X are complex polynomials in the canonical class k_X and the quadratic form q_X [FMM]. In the $SO(3)$-case one finds the following result:

THEOREM 4. *Let X be a simply connected algebraic surface with $p_g(X) > 0, w_2(X) \neq 0$, and with a big monodromy group. Then, for a principal $SO(3)$-bundle P,*

$$\Phi_{k_X,\alpha,P}(X) \in \mathbf{C}[k_X, q_X] .$$

COROLLARY 5. *Let X be a simply connected algebraic surface with $p_g(X) > 0$ and with a big monodromy group. Then $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$, if k_X divides a nontrivial polynomial invariant.*

The corollary follows from the fact that if k_X divides a nontrivial polynomial invariant, then it is its only linear factor up to multiples (cf. [FMM]).

When are the assumptions of Corollary 5 satisfied? It follows from Theorem 1 that the surfaces listed in this theorem have big monodromy.

Let X be any simply connected algebraic surface with a big monodromy group. If $p_g(X) \equiv 0 \pmod 2$ then the degree of $\Phi_l(X)$ is odd. If $p_g(X) \equiv 1 \pmod 2$ and $k_X^2 \equiv 1 \pmod 2$ then the degree of $\Phi_{k_X,\alpha,P}(X)$ is odd. So k_X divides $\Phi_l(X)$ or $\Phi_{k_X,\alpha,P}(X)$ in these cases.

Remark. Theorem 4 and its corollary remain true for polynomials $\Phi_{c,\alpha,P}(X)$ if $c \in H^2(X, \mathbf{Z})$ is a class with $\bar{c} \neq 0$ such that $\overline{f^*(c)} = \bar{c}$ for all $f \in \psi(\mathrm{Diff}_+(X))$. The question which elements of $H^2(X, \mathbf{Z})$ or $H^2(X, \mathbf{Z}/2)$ have this invariance property will be treated in §4.

3. Non-realizable isometries

We shall show that for a simply connected algebraic surface with odd geometric genus, -1 is not induced by an orientation preserving diffeomorphism. For K3 surfaces this was shown by Donaldson in the proof of [D, Proposition 6.2]. There he proves the nontriviality of a certain polynomial $\Phi_{c,\alpha,P}(X)$ for a K3 surface X. With Zuo's nontriviality result (Theorem 3) we are able to generalize this as follows.

Theorem 6. *If X is a simply connected algebraic surface with $p_g(X) \equiv 1 \pmod 2$ then $-1 \notin \psi(\mathrm{Diff}_+(X))$.*

Proof. Suppose that there is an orientation preserving diffeomorphism $f: X \to X$ such that $f^* = -1$. Let $c \in H^{1,1}(X, \mathbf{Z})$ be a class with $\bar{c} \neq 0$, and choose a principal $SO(3)$-bundle P with $w_2(P) = \bar{c}$ such that $\Phi_{c,\alpha,P}(X)$ is nontrivial. This is possible according to Theorem 3. Then

$$f^*\Phi_{c,\alpha,P}(X) = (-1)^d \Phi_{c,\alpha,P}(X) ,$$

since $\Phi_{c,\alpha,P}(X)$ is a polynomial of degree d on L.

On the other hand, by §2(c)

$$f^*\Phi_{c,\alpha,P}(X) = \Phi_{f^*c, f^*\alpha, f^*P}(X) .$$

We have $f^*c = -c$ and $f^*\alpha = -\alpha$ because $f^* = -1$ and the dimension of α is odd. Since f is orientation preserving and $f^* = -1$ we find $f^*p_1(P) = p_1(P)$ and $f^*w_2(P) = w_2(P)$, so that the bundle f^*P is isomorphic to P. Therefore