

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	37 (1991)
<b>Heft:</b>	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
 <b>Artikel:</b>	ON THE DIFFEOMORPHISM GROUPS OF CERTAIN ALGEBRAIC SURFACES
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<b>Kapitel:</b>	1. Construction of diffeomorphisms
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-58742">https://doi.org/10.5169/seals-58742</a>

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The first author is grateful to the Department of Mathematics and Computing Science of the Eindhoven University of Technology, Eindhoven, The Netherlands, where he did the work on this paper. The second author likes to thank the IHES for its hospitality and support. We both thank J. Wahl for a very useful remark concerning Theorem 12.

## 1. CONSTRUCTION OF DIFFEOMORPHISMS

Let  $X$  be a simply connected smooth compact complex algebraic surface with quadratic intersection form  $q_X: H_2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$  and canonical class  $k_X \in H^2(X, \mathbf{Z}) = \text{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$ . The second homology group  $H_2(X, \mathbf{Z})$  endowed with the quadratic form  $q_X$  forms a lattice which we denote by  $L$ . Let  $O(L)$  be the corresponding group of isometries.

We introduce some notation: Let  $b_2^+(X)$  denote the dimension of any maximal subspace of  $L_{\mathbf{R}} = L \otimes \mathbf{R}$  on which  $q_X$  is positive definite. Note that  $b_2^+(X) = 2p_g(X) + 1$  where  $p_g(X)$  is the geometric genus of  $X$ . The set of all oriented maximal positive definite subspaces of  $L_{\mathbf{R}}$  forms an open subset  $\Omega$  of the Grassmannian  $G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$  of oriented  $b_2^+(X)$ -dimensional subspaces of the vector space  $L_{\mathbf{R}}$ . It has two components if  $q_X$  is indefinite. We define  $O'(L)$  to be the subgroup of  $O(L)$  consisting of those automorphisms which leave each component of  $\Omega$  invariant. (For an equivalent definition of  $O'(L)$  see [E2, 4.1].) Let  $O_k(L)$  be the subgroup of  $O(L)$  consisting of automorphisms preserving  $k = k_X$ . Finally we define  $O'_k(L) := O_k(L) \cap O'(L)$ .

An important subset of  $\text{Diff}_+(X)$  is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing  $X$  as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping  $\pi: \mathcal{X} \rightarrow T$  of connected complex spaces  $\mathcal{X}$  and  $T$ ;  $\pi$  is the projection of a locally trivial differentiable fibre bundle, so that for a point  $t_0 \in T$  with  $X = \pi^{-1}(t_0)$  there is a *monodromy representation*  $\rho: \pi_1(T, t_0) \rightarrow \text{Diff}_+(X)$ . The image  $\Gamma$  of  $\psi \circ \rho$  in  $O(L)$  is called the *monodromy group* of the smooth family. The monodromy group preserves  $k_X$ . It also preserves the components of  $\Omega$ : to see this consider a loop  $\tau$  representing an element in  $\pi_1(T, t_0)$  and let  $g_t: X_{t_0} = X \rightarrow X_t$  be the diffeomorphisms corresponding to  $t \in \tau$ . The mapping  $t \mapsto (g_t)_*(\alpha) \in G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$  is continuous for every  $\alpha \in \Omega$ , hence  $\Gamma \subset O'_k(L)$ .

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group  $O'_k(L)$ . This is summarized in the

following theorem. Among these surfaces there are certain iterated ramified coverings of  $\mathbf{P}^1 \times \mathbf{P}^1$  or  $\mathbf{P}^2$  which were studied by Moishezon [M] and Salvetti [S]: Let  $(n_1, \dots, n_r)$  be a sequence of positive integers. For Moishezon's construction, let  $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$  and let  $C \subset \mathbf{P}^1 \times \mathbf{P}^1$  be the divisor  $\{\text{pt}\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\text{pt}\}$ . Construct a sequence  $\beta_i: X_i \rightarrow X_{i-1}$  of cyclic coverings  $\beta_i$  of degree 3 with nonsingular branch locus linearly equivalent to  $(\beta_{i-1} \circ \dots \circ \beta_1)^*(3n_i C)$ . Let  $X_r = X_r(n_1, \dots, n_r)$ . We call a surface  $X_r(n_1, \dots, n_r)$  with  $n_i \geq 2$  for some  $i$ ,  $1 \leq i \leq r$ , (cf. [M, §4, Remark 3]) a *Moishezon surface*. For Salvetti's construction we need in addition to  $(n_1, \dots, n_r)$  a sequence  $(d_1, \dots, d_r)$  of positive integers satisfying  $d_i \mid n_i$  for all  $i = 1, \dots, r$ . Let  $Y_0 = \mathbf{P}^2$ , and choose smooth curves  $C_i \subset \mathbf{P}^2$  of degree  $n_i$ ,  $i = 1, \dots, r$ , so that  $C := C_1 \cup \dots \cup C_r$  has only normal crossings. Construct a sequence  $\beta_i: Y_i \rightarrow Y_{i-1}$  of cyclic coverings  $\beta_i$  of degree  $d_i$  ramified over  $(\beta_{i-1} \circ \dots \circ \beta_1)^*(C_i)$ . A *Salvetti surface* is a surface  $Y_r = Y_r(n_1, \dots, n_r; d_1, \dots, d_r)$ , if at least one  $n_i > 5$  with the corresponding  $d_i \geq 3$  (cf. [S, Corollary to Proposition 2]).

**THEOREM 1.** *Let  $X$  be a simply connected algebraic surface with  $p_g(X) > 0$ , and suppose that  $X$  is either*

- (i) *a complete intersection, or*
- (ii) *a Moishezon or Salvetti surface.*

*Then there is a smooth family  $\pi: \mathcal{X} \rightarrow T$  with  $\pi^{-1}(t_0) = X$  for some  $t_0 \in T$  with monodromy group*

$$\Gamma = O'_k(L) .$$

For complete intersection surfaces this is proved in [B, E2, E3], for Moishezon surfaces see [M], and for Salvetti surfaces see [S], together with [EO, Theorem 2.5 and §1].

Next we construct an orientation preserving diffeomorphism  $\sigma: X \rightarrow X$  with  $\sigma^*(k_X) = -k_X$ . Suppose  $X$  is embedded in a complex projective space  $\mathbf{P}^N$  so that  $X$  is the zero locus of a finite set  $\{f_1, \dots, f_M\}$  of homogeneous polynomials in the coordinates  $z_0, \dots, z_N$  of  $\mathbf{P}^N$ .

Denote by  $\sigma: \mathbf{C} \rightarrow \mathbf{C}$  complex conjugation, and let  $X^\sigma \subset \mathbf{P}^N$  be the zero locus of  $\{f_1^\sigma, \dots, f_M^\sigma\}$ , where  $f_i^\sigma$  is obtained from  $f_i$  by applying  $\sigma$  to all the coefficients. Then  $\sigma: \mathbf{P}^N \rightarrow \mathbf{P}^N$  induces a diffeomorphism  $X \rightarrow X^\sigma$ , also denoted by  $\sigma$ , which satisfies  $\sigma^*(k_{X^\sigma}) = -k_X$ . If  $X$  is given by real equations  $f_1, \dots, f_M$ , then  $\sigma: X \rightarrow X^\sigma = X$  is a self-diffeomorphism of  $X$ . Such equations can be found if  $X$  is a complete intersection and if  $X$  is a

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let  $X$  be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \text{id}\} \subset \psi(\text{Diff}_+(X)) .$$

## 2. INVARIANCE OF THE CANONICAL CLASS

S. K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces  $X$  with  $p_g(X) > 0$ . We assume from now on that  $X$  is such a surface. There are two types of invariants according to the gauge group being  $SU(2)$  or  $SO(3)$ .

Let us first recall the  $SU(2)$ -case. Principal  $SU(2)$ -bundles over  $X$  are classified by their second Chern class  $c_2(P)$ . For each  $l > l_0$ , using such a bundle with  $c_2(P) = l$ , Donaldson defines a polynomial

$$\Phi_l(X) : \text{Sym}^d(L) \rightarrow \mathbf{Z}$$

of degree  $d = d(l) = 4l - 3(p_g(X) + 1)$ , which depends only on the underlying  $C^\infty$ -structure of  $X$  and is invariant up to sign under  $\psi(\text{Diff}_+(X))$ . Donaldson shows that these invariants are nontrivial for all sufficiently large  $l$  [D].

We will need the slightly more complicated  $SO(3)$ -invariants. The simple Lie group  $SO(3)$  is isomorphic to  $PU(2)$ , so that one has an exact sequence

$$1 \rightarrow S^1 \rightarrow U(2) \rightarrow SO(3) \rightarrow 1 .$$

Let  $P$  be a principal  $SO(3)$ -bundle over  $X$ . Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class  $w_2(P) \in H^2(X, \mathbf{Z}/2)$  and the first Pontryagin class  $p_1(P) \in H^4(X, \mathbf{Z})$ .

Suppose that  $w_2(P)$  is nonzero and choose an integral lifting  $c$  of  $w_2(P)$ , i.e.  $c \in H^2(X, \mathbf{Z})$ ,  $\bar{c} = w_2(P)$  (here  $\bar{c}$  means the reduction of  $c$  modulo 2). Such a lifting exists since  $X$  is simply connected, and determines a  $U(2)$ -lifting  $\hat{P}$  of  $P$ , i.e. a  $U(2)$ -bundle  $\hat{P}$  with  $\hat{P}/S^1 = P$  and with  $c = c_1(\hat{P})$  [HH]. The Chern classes of  $\hat{P}$  are related to the characteristic classes of  $P$  by  $w_2(P) = \bar{c}_1(\hat{P})$  and  $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$ . In addition to this choose an element  $\alpha \in \Omega$ . Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c, \alpha, P}(X) : \text{Sym}^d(L) \rightarrow \mathbf{Z}$$