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1. CONSTRUCTION OF DIFFEOMORPHISMS

Let X be a simply connected smooth compact complex algebraic surface with quadratic intersection form $q_X: H_2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$ and canonical class $k_X \in H^2(X, \mathbf{Z}) = \text{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$. The second homology group $H_2(X, \mathbf{Z})$ endowed with the quadratic form q_X forms a lattice which we denote by L . Let $O(L)$ be the corresponding group of isometries.

We introduce some notation: Let $b_2^+(X)$ denote the dimension of any maximal subspace of $L_{\mathbf{R}} = L \otimes \mathbf{R}$ on which q_X is positive definite. Note that $b_2^+(X) = 2p_g(X) + 1$ where $p_g(X)$ is the geometric genus of X . The set of all oriented maximal positive definite subspaces of $L_{\mathbf{R}}$ forms an open subset Ω of the Grassmannian $G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$ of oriented $b_2^+(X)$ -dimensional subspaces of the vector space $L_{\mathbf{R}}$. It has two components if q_X is indefinite. We define $O'(L)$ to be the subgroup of $O(L)$ consisting of those automorphisms which leave each component of Ω invariant. (For an equivalent definition of $O'(L)$ see [E2, 4.1].) Let $O_k(L)$ be the subgroup of $O(L)$ consisting of automorphisms preserving $k = k_X$. Finally we define $O'_k(L) := O_k(L) \cap O'(L)$.

An important subset of $\text{Diff}_+(X)$ is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing X as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping $\pi: \mathcal{X} \rightarrow T$ of connected complex spaces \mathcal{X} and T ; π is the projection of a locally trivial differentiable fibre bundle, so that for a point $t_0 \in T$ with $X = \pi^{-1}(t_0)$ there is a *monodromy representation* $\rho: \pi_1(T, t_0) \rightarrow \text{Diff}_+(X)$. The image Γ of $\psi \circ \rho$ in $O(L)$ is called the *monodromy group* of the smooth family. The monodromy group preserves k_X . It also preserves the components of Ω : to see this consider a loop τ representing an element in $\pi_1(T, t_0)$ and let $g_t: X_{t_0} = X \rightarrow X_t$ be the diffeomorphisms corresponding to $t \in \tau$. The mapping $t \mapsto (g_t)_*(\alpha) \in G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$ is continuous for every $\alpha \in \Omega$, hence $\Gamma \subset O'_k(L)$.

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group $O'_k(L)$. This is summarized in the

following theorem. Among these surfaces there are certain iterated ramified coverings of $\mathbf{P}^1 \times \mathbf{P}^1$ or \mathbf{P}^2 which were studied by Moishezon [M] and Salvetti [S]: Let (n_1, \dots, n_r) be a sequence of positive integers. For Moishezon's construction, let $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the divisor $\{\text{pt}\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\text{pt}\}$. Construct a sequence $\beta_i: X_i \rightarrow X_{i-1}$ of cyclic coverings β_i of degree 3 with nonsingular branch locus linearly equivalent to $(\beta_{i-1} \circ \dots \circ \beta_1)^*(3n_i C)$. Let $X_r = X_r(n_1, \dots, n_r)$. We call a surface $X_r(n_1, \dots, n_r)$ with $n_i \geq 2$ for some i , $1 \leq i \leq r$, (cf. [M, §4, Remark 3]) a *Moishezon surface*. For Salvetti's construction we need in addition to (n_1, \dots, n_r) a sequence (d_1, \dots, d_r) of positive integers satisfying $d_i \mid n_i$ for all $i = 1, \dots, r$. Let $Y_0 = \mathbf{P}^2$, and choose smooth curves $C_i \subset \mathbf{P}^2$ of degree n_i , $i = 1, \dots, r$, so that $C := C_1 \cup \dots \cup C_r$ has only normal crossings. Construct a sequence $\beta_i: Y_i \rightarrow Y_{i-1}$ of cyclic coverings β_i of degree d_i ramified over $(\beta_{i-1} \circ \dots \circ \beta_1)^*(C_i)$. A *Salvetti surface* is a surface $Y_r = Y_r(n_1, \dots, n_r; d_1, \dots, d_r)$, if at least one $n_i > 5$ with the corresponding $d_i \geq 3$ (cf. [S, Corollary to Proposition 2]).

THEOREM 1. *Let X be a simply connected algebraic surface with $p_g(X) > 0$, and suppose that X is either*

- (i) *a complete intersection, or*
- (ii) *a Moishezon or Salvetti surface.*

Then there is a smooth family $\pi: \mathcal{X} \rightarrow T$ with $\pi^{-1}(t_0) = X$ for some $t_0 \in T$ with monodromy group

$$\Gamma = O'_k(L) .$$

For complete intersection surfaces this is proved in [B, E2, E3], for Moishezon surfaces see [M], and for Salvetti surfaces see [S], together with [EO, Theorem 2.5 and §1].

Next we construct an orientation preserving diffeomorphism $\sigma: X \rightarrow X$ with $\sigma^*(k_X) = -k_X$. Suppose X is embedded in a complex projective space \mathbf{P}^N so that X is the zero locus of a finite set $\{f_1, \dots, f_M\}$ of homogeneous polynomials in the coordinates z_0, \dots, z_N of \mathbf{P}^N .

Denote by $\sigma: \mathbf{C} \rightarrow \mathbf{C}$ complex conjugation, and let $X^\sigma \subset \mathbf{P}^N$ be the zero locus of $\{f_1^\sigma, \dots, f_M^\sigma\}$, where f_i^σ is obtained from f_i by applying σ to all the coefficients. Then $\sigma: \mathbf{P}^N \rightarrow \mathbf{P}^N$ induces a diffeomorphism $X \rightarrow X^\sigma$, also denoted by σ , which satisfies $\sigma^*(k_{X^\sigma}) = -k_X$. If X is given by real equations f_1, \dots, f_M , then $\sigma: X \rightarrow X^\sigma = X$ is a self-diffeomorphism of X . Such equations can be found if X is a complete intersection and if X is a

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let X be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \text{id}\} \subset \psi(\text{Diff}_+(X)) .$$

2. INVARIANCE OF THE CANONICAL CLASS

S. K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces X with $p_g(X) > 0$. We assume from now on that X is such a surface. There are two types of invariants according to the gauge group being $SU(2)$ or $SO(3)$.

Let us first recall the $SU(2)$ -case. Principal $SU(2)$ -bundles over X are classified by their second Chern class $c_2(P)$. For each $l > l_0$, using such a bundle with $c_2(P) = l$, Donaldson defines a polynomial

$$\Phi_l(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$

of degree $d = d(l) = 4l - 3(p_g(X) + 1)$, which depends only on the underlying C^∞ -structure of X and is invariant up to sign under $\psi(\text{Diff}_+(X))$. Donaldson shows that these invariants are nontrivial for all sufficiently large l [D].

We will need the slightly more complicated $SO(3)$ -invariants. The simple Lie group $SO(3)$ is isomorphic to $PU(2)$, so that one has an exact sequence

$$1 \rightarrow S^1 \rightarrow U(2) \rightarrow SO(3) \rightarrow 1 .$$

Let P be a principal $SO(3)$ -bundle over X . Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbf{Z}/2)$ and the first Pontryagin class $p_1(P) \in H^4(X, \mathbf{Z})$.

Suppose that $w_2(P)$ is nonzero and choose an integral lifting c of $w_2(P)$, i.e. $c \in H^2(X, \mathbf{Z})$, $\bar{c} = w_2(P)$ (here \bar{c} means the reduction of c modulo 2). Such a lifting exists since X is simply connected, and determines a $U(2)$ -lifting \hat{P} of P , i.e. a $U(2)$ -bundle \hat{P} with $\hat{P}/S^1 = P$ and with $c = c_1(\hat{P})$ [HH]. The Chern classes of \hat{P} are related to the characteristic classes of P by $w_2(P) = \bar{c}_1(\hat{P})$ and $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$. In addition to this choose an element $\alpha \in \Omega$. Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c, \alpha, P}(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$