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## 1. CONSTRUCTION OF DIFFEOMORPHISMS

Let  $X$  be a simply connected smooth compact complex algebraic surface with quadratic intersection form  $q_X: H_2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$  and canonical class  $k_X \in H^2(X, \mathbf{Z}) = \text{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$ . The second homology group  $H_2(X, \mathbf{Z})$  endowed with the quadratic form  $q_X$  forms a lattice which we denote by  $L$ . Let  $O(L)$  be the corresponding group of isometries.

We introduce some notation: Let  $b_2^+(X)$  denote the dimension of any maximal subspace of  $L_{\mathbf{R}} = L \otimes \mathbf{R}$  on which  $q_X$  is positive definite. Note that  $b_2^+(X) = 2p_g(X) + 1$  where  $p_g(X)$  is the geometric genus of  $X$ . The set of all oriented maximal positive definite subspaces of  $L_{\mathbf{R}}$  forms an open subset  $\Omega$  of the Grassmannian  $G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$  of oriented  $b_2^+(X)$ -dimensional subspaces of the vector space  $L_{\mathbf{R}}$ . It has two components if  $q_X$  is indefinite. We define  $O'(L)$  to be the subgroup of  $O(L)$  consisting of those automorphisms which leave each component of  $\Omega$  invariant. (For an equivalent definition of  $O'(L)$  see [E2, 4.1].) Let  $O_k(L)$  be the subgroup of  $O(L)$  consisting of automorphisms preserving  $k = k_X$ . Finally we define  $O'_k(L) := O_k(L) \cap O'(L)$ .

An important subset of  $\text{Diff}_+(X)$  is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing  $X$  as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping  $\pi: \mathcal{X} \rightarrow T$  of connected complex spaces  $\mathcal{X}$  and  $T$ ;  $\pi$  is the projection of a locally trivial differentiable fibre bundle, so that for a point  $t_0 \in T$  with  $X = \pi^{-1}(t_0)$  there is a *monodromy representation*  $\rho: \pi_1(T, t_0) \rightarrow \text{Diff}_+(X)$ . The image  $\Gamma$  of  $\psi \circ \rho$  in  $O(L)$  is called the *monodromy group* of the smooth family. The monodromy group preserves  $k_X$ . It also preserves the components of  $\Omega$ : to see this consider a loop  $\tau$  representing an element in  $\pi_1(T, t_0)$  and let  $g_t: X_{t_0} = X \rightarrow X_t$  be the diffeomorphisms corresponding to  $t \in \tau$ . The mapping  $t \mapsto (g_t)_*(\alpha) \in G^{\text{or}}(b_2^+(X), L_{\mathbf{R}})$  is continuous for every  $\alpha \in \Omega$ , hence  $\Gamma \subset O'_k(L)$ .

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group  $O'_k(L)$ . This is summarized in the

following theorem. Among these surfaces there are certain iterated ramified coverings of  $\mathbf{P}^1 \times \mathbf{P}^1$  or  $\mathbf{P}^2$  which were studied by Moishezon [M] and Salvetti [S]: Let  $(n_1, \dots, n_r)$  be a sequence of positive integers. For Moishezon's construction, let  $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$  and let  $C \subset \mathbf{P}^1 \times \mathbf{P}^1$  be the divisor  $\{\text{pt}\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\text{pt}\}$ . Construct a sequence  $\beta_i: X_i \rightarrow X_{i-1}$  of cyclic coverings  $\beta_i$  of degree 3 with nonsingular branch locus linearly equivalent to  $(\beta_{i-1} \circ \dots \circ \beta_1)^*(3n_i C)$ . Let  $X_r = X_r(n_1, \dots, n_r)$ . We call a surface  $X_r(n_1, \dots, n_r)$  with  $n_i \geq 2$  for some  $i$ ,  $1 \leq i \leq r$ , (cf. [M, §4, Remark 3]) a *Moishezon surface*. For Salvetti's construction we need in addition to  $(n_1, \dots, n_r)$  a sequence  $(d_1, \dots, d_r)$  of positive integers satisfying  $d_i | n_i$  for all  $i = 1, \dots, r$ . Let  $Y_0 = \mathbf{P}^2$ , and choose smooth curves  $C_i \subset \mathbf{P}^2$  of degree  $n_i$ ,  $i = 1, \dots, r$ , so that  $C := C_1 \cup \dots \cup C_r$  has only normal crossings. Construct a sequence  $\beta_i: Y_i \rightarrow Y_{i-1}$  of cyclic coverings  $\beta_i$  of degree  $d_i$  ramified over  $(\beta_{i-1} \circ \dots \circ \beta_1)^*(C_i)$ . A *Salvetti surface* is a surface  $Y_r = Y_r(n_1, \dots, n_r; d_1, \dots, d_r)$ , if at least one  $n_i > 5$  with the corresponding  $d_i \geq 3$  (cf. [S, Corollary to Proposition 2]).

**THEOREM 1.** *Let  $X$  be a simply connected algebraic surface with  $p_g(X) > 0$ , and suppose that  $X$  is either*

- (i) *a complete intersection, or*
- (ii) *a Moishezon or Salvetti surface.*

*Then there is a smooth family  $\pi: \mathcal{X} \rightarrow T$  with  $\pi^{-1}(t_0) = X$  for some  $t_0 \in T$  with monodromy group*

$$\Gamma = O'_k(L) .$$

For complete intersection surfaces this is proved in [B, E2, E3], for Moishezon surfaces see [M], and for Salvetti surfaces see [S], together with [EO, Theorem 2.5 and §1].

Next we construct an orientation preserving diffeomorphism  $\sigma: X \rightarrow X$  with  $\sigma^*(k_X) = -k_X$ . Suppose  $X$  is embedded in a complex projective space  $\mathbf{P}^N$  so that  $X$  is the zero locus of a finite set  $\{f_1, \dots, f_M\}$  of homogeneous polynomials in the coordinates  $z_0, \dots, z_N$  of  $\mathbf{P}^N$ .

Denote by  $\sigma: \mathbf{C} \rightarrow \mathbf{C}$  complex conjugation, and let  $X^\sigma \subset \mathbf{P}^N$  be the zero locus of  $\{f_1^\sigma, \dots, f_M^\sigma\}$ , where  $f_i^\sigma$  is obtained from  $f_i$  by applying  $\sigma$  to all the coefficients. Then  $\sigma: \mathbf{P}^N \rightarrow \mathbf{P}^N$  induces a diffeomorphism  $X \rightarrow X^\sigma$ , also denoted by  $\sigma$ , which satisfies  $\sigma^*(k_{X^\sigma}) = -k_X$ . If  $X$  is given by real equations  $f_1, \dots, f_M$ , then  $\sigma: X \rightarrow X^\sigma = X$  is a self-diffeomorphism of  $X$ . Such equations can be found if  $X$  is a complete intersection and if  $X$  is a

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let  $X$  be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \text{id}\} \subset \psi(\text{Diff}_+(X)) .$$

## 2. INVARIANCE OF THE CANONICAL CLASS

S. K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces  $X$  with  $p_g(X) > 0$ . We assume from now on that  $X$  is such a surface. There are two types of invariants according to the gauge group being  $SU(2)$  or  $SO(3)$ .

Let us first recall the  $SU(2)$ -case. Principal  $SU(2)$ -bundles over  $X$  are classified by their second Chern class  $c_2(P)$ . For each  $l > l_0$ , using such a bundle with  $c_2(P) = l$ , Donaldson defines a polynomial

$$\Phi_l(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$

of degree  $d = d(l) = 4l - 3(p_g(X) + 1)$ , which depends only on the underlying  $C^\infty$ -structure of  $X$  and is invariant up to sign under  $\psi(\text{Diff}_+(X))$ . Donaldson shows that these invariants are nontrivial for all sufficiently large  $l$  [D].

We will need the slightly more complicated  $SO(3)$ -invariants. The simple Lie group  $SO(3)$  is isomorphic to  $PU(2)$ , so that one has an exact sequence

$$1 \rightarrow S^1 \rightarrow U(2) \rightarrow SO(3) \rightarrow 1 .$$

Let  $P$  be a principal  $SO(3)$ -bundle over  $X$ . Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class  $w_2(P) \in H^2(X, \mathbf{Z}/2)$  and the first Pontryagin class  $p_1(P) \in H^4(X, \mathbf{Z})$ .

Suppose that  $w_2(P)$  is nonzero and choose an integral lifting  $c$  of  $w_2(P)$ , i.e.  $c \in H^2(X, \mathbf{Z})$ ,  $\bar{c} = w_2(P)$  (here  $\bar{c}$  means the reduction of  $c$  modulo 2). Such a lifting exists since  $X$  is simply connected, and determines a  $U(2)$ -lifting  $\hat{P}$  of  $P$ , i.e. a  $U(2)$ -bundle  $\hat{P}$  with  $\hat{P}/S^1 = P$  and with  $c = c_1(\hat{P})$  [HH]. The Chern classes of  $\hat{P}$  are related to the characteristic classes of  $P$  by  $w_2(P) = \bar{c}_1(\hat{P})$  and  $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$ . In addition to this choose an element  $\alpha \in \Omega$ . Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c, \alpha, P}(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$