$$\textstyle\sum_i \delta(V_i) = f_0(P) - s_0(P)$$

$$\textstyle\sum_j \delta(E_j) = f_1(P) - s_1(P) ,$$

so (3) becomes

$$(f_0(P) - \textstyle\sum_i \delta(V_i)) - (f_1(P) - \sum_j \delta(E_j)) + f_2(P) = f_3(P)$$

or,

$$\textstyle\sum_i \delta(V_i) - \sum_j \delta(E_j) = f_0(P) - f_1(P) + f_2(P) - f_3(P) = \chi(P)$$

which is (1), and the theorem is proved.

An exactly analogous argument holds in n dimensions (for all $n \geqslant 3$). In the proof we use the $(n-1)$-dimensional form of Gram's Theorem (see [3], [7]) for each $(n-1)$-face F_k of P:

$$s_0(F_k) - s_1(F_k) + s_2(F_k) - \dots + (-1)^{n-2} s_{n-2}(F_k) = (-1)^n$$

with a notation analogous to that used in (2). This leads to the statement:

DESCARTES' THEOREM IN n DIMENSIONS. *Let P be an elementary polytope in E^n, let $\delta(F_i^m) = 1 - s(F_i^m)$ be the deficiency of P at the m-face F_i^m of P $(m = 0, 1, \dots, n-3)$, and let $\delta_m(P) = \sum_i \delta(F_i^m)$, where summation is over all the m-faces F_i^m of P. Then*

$$\sum_{m=0}^{n-3} (-1)^m \delta_m(P) = \chi(P) ,$$

where $\chi(P)$ is the Euler characteristic of P.

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