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## §2. $L$ -FUNCTIONS

Throughout this section,  $V$  denotes a *monic* polynomial over  $GF(q)$ , and  $v$  ranges over the distinct monic irreducible factors of  $V$  over  $GF(q)$ . Write

$$(2.1) \quad V = \prod_{v|V} v^{\text{ord}_v V}, \quad F = F_V = \prod_{v|V} v.$$

If no exponent  $\text{ord}_v V$  in (2.1) is divisible by  $q - 1$ , then  $V$  is said to be *primitive*. Note that  $V = 1$  is primitive. For any monic polynomial

$$(2.2) \quad W = W(x) = x^n + w_{n-1}x^{n-1} + w_{n-2}x^{n-2} + \cdots + w_0$$

over  $GF(q)$ , set

$$(2.3) \quad \alpha(W) = w_{n-1}, \quad \beta(W) = w_{n-1}^2/2 - w_{n-2}.$$

Define the  $L$ -functions

$$(2.4) \quad L(t, V) = \sum_W \tau(R(V, W)) t^{\deg W},$$

$$(2.4a) \quad L_1(t, V) = \sum_W \psi(\alpha(W)) \tau(R(V, W)) t^{\deg W},$$

$$(2.4b) \quad L_2(t, V) = \sum_W \psi(\beta(W)) \tau(R(V, W)) t^{\deg W},$$

where in each sum,  $W$  ranges over all monic polynomials over  $GF(q)$ , and  $R(V, W)$  is the resultant of  $V$  and  $W$ . It is easily checked that

$$(2.5) \quad \begin{aligned} L(t, 1) &= (1 - qt)^{-1}, & L_1(t, 1) &= 1, \\ L_2(t, 1) &= 1 + \phi(2)G((q-1)/2)t. \end{aligned}$$

Since the summands in (2.4), (2.4a), (2.4b) are multiplicative in  $W$ , each of the  $L$ -functions has an Euler product expansion. Thus we have the following result.

LEMMA 2.1. Write  $V = GH$  where  $G$  and  $H$  are monic, relatively prime polynomials over  $GF(q)$  with  $G$  primitive and  $H$  a  $(q-1)$ th power. Then

$$(2.6) \quad L(t, V) = L(t, G) \prod_{v|H} (1 - \tau(R(G, v)) t^{\deg v}),$$

$$(2.6a) \quad L_1(t, V) = L_1(t, G) \prod_{v|H} (1 - \psi(\alpha(v)) \tau(R(G, v)) t^{\deg v}),$$

and

$$(2.6b) \quad L_2(t, V) = L_2(t, G) \prod_{v|H} (1 - \psi(\beta(v))\tau(R(G, v))t^{\deg v}) .$$

The next lemma evaluates certain generating functions defined in terms of the function  $L$  (but not  $L_1$  or  $L_2$ ).

LEMMA 2.2. For all integers  $a, b > 0$ ,

$$(2.7) \quad \sum_W \tau(W^a(0)W^b(1))L(t, W^{q-1})z^{\deg W} = \begin{cases} \frac{1 + \tau(-1)^a J(a, b)z}{(1-qt)(1 + \tau(-1)^a J(a, b)zt)}, & \text{if } a \not\equiv 0 \pmod{q-1} \\ & \text{or } b \not\equiv 0 \pmod{q-1}, \\ \frac{(1-z)^2(1-qzt)}{(1-qt)(1-qz)(1-zt)^2}, & \text{if } a \equiv b \equiv 0 \pmod{q-1}, \end{cases}$$

$$(2.7a) \quad \sum_W \psi(\alpha(W^b))\tau(W(0)^a)L(t, W^{q-1})z^{\deg W} = \frac{1 + \bar{\tau}^a(b)G(a)z}{(1-qt)(1 + \bar{\tau}^a(b)G(a)zt)},$$

and

$$(2.7b) \quad \sum_W \psi(\beta(W^b))L(t, W^{q-1})z^{\deg W} = \frac{1 + \phi(2b)G((q-1)/2)z}{(1-qt)(1 + \phi(2b)G((q-1)/2)zt)},$$

where in each sum,  $W$  ranges over all monic polynomials over  $GF(q)$  and  $\alpha, \beta$  are as defined in (2.3).

*Proof.* Fix monic  $V = V(x)$  and let  $w$  range over monic irreducibles over  $GF(q)$ . By (2.6),

$$\begin{aligned} & \sum_W \tau(R(V, W))L(t, W^{q-1})z^{\deg W} \\ &= L(t, 1) \sum_W z^{\deg W} \tau(R(V, W)) \prod_{w|W} (1 - t^{\deg w}) \\ &= L(t, 1) \sum_W \prod_{w|W} \{(1 - t^{\deg w}) (\tau(R(V, w))z^{\deg w})^{\text{ord}_w W}\} \\ &= L(t, 1) \prod_w \left\{ 1 + (1 - t^{\deg w}) \sum_{m=1}^{\infty} (\tau(R(V, w))z^{\deg w})^m \right\} \\ &= L(t, 1) \prod_w \frac{1 - \tau(R(V, w)) (zt)^{\deg w}}{1 - \tau(R(V, w))z^{\deg w}} = \frac{L(t, 1)L(z, V)}{L(zt, V)}. \end{aligned}$$

Taking  $V = x^a(x-1)^b$ , we easily deduce (2.7). The proofs of (2.7a) and (2.7b) are similar.  $\square$

It is shown in [1, Prop. 2.1] that if  $V$  is primitive of degree  $> 0$ , then  $L(t, V)$  is a polynomial in  $t$  of degree  $(\deg F - 1)$  with leading coefficient

$$(2.8) \quad \varepsilon(V) = \sigma(F)\tau(R(V, F'))G^*(\deg V)^{-1} \prod_{v|F} G^*(\text{ord}_v V)^{\deg v},$$

where

$$G^*(a) := q/G(-a).$$

By (2.6), if  $V$  is a  $(q-1)$ th power, then

$$L(t, V) = (1 - qt)^{-1} \prod_{v|V} (1 - t^{\deg v}),$$

but otherwise  $L(t, V)$  is a polynomial of degree  $(\deg F - 1)$ . The following lemma shows that for all  $V$ ,  $L_1(t, V)$  and  $L_2(t, V)$  are polynomials of degrees  $\deg F$  and  $\deg F + 1$ , respectively. Moreover, for primitive  $V \neq 1$ , the coefficient  $\varepsilon_1(V)$  of  $t^{\deg F}$  in  $L_1(t, V)$  and the coefficient  $\varepsilon_2(V)$  of  $t^{1+\deg F}$  in  $L_2(t, V)$  are given explicitly.

**LEMMA 2.3.** *For each monic polynomial  $V$  over  $GF(q)$ ,  $L_1(t, V)$  and  $L_2(t, V)$  are polynomials in  $t$  of degrees  $\deg F$  and  $1 + \deg F$ , respectively. If moreover  $V \neq 1$  is primitive, the leading coefficients of  $L_1(t, V)$  and  $L_2(t, V)$  are given by*

$$(2.8a) \quad \varepsilon_1(V) = \psi(\alpha(F))\sigma(F)\tau(R(V, -F')) \prod_{v|F} G^*(\text{ord}_v V)^{\deg v},$$

and

$$(2.8b) \quad \varepsilon_2(V) = \phi(2)G((q-1)/2)\psi(\beta(F))\sigma(F)\tau(R(V, F')) \prod_{v|F} G^*(\text{ord}_v V)^{\deg v},$$

respectively, where  $G^*(a) = q/G(-a)$ .

*Proof.* Fix an integer  $m > \deg F$  and fix  $\alpha \in GF(q)$ . Since  $m > \deg F$ , it is not hard to see that the monic polynomials  $W$  over  $GF(q)$  of degree  $m$  with  $\alpha(W) = \alpha$  run through each residue class modulo  $F$  exactly  $q^{m-1-\deg F}$  times. Since  $R(V, W)$  depends only on the residue class of  $W$  modulo  $F$ , the coefficient of  $t^m$  in  $L_1(t, V)$  thus equals

$$\begin{aligned} & \sum_{\substack{W \text{ monic} \\ \deg W = m}} \psi(\alpha(W)) \tau(R(V, W)) \\ &= q^{m-1-\deg F} \sum_{\substack{U \\ \deg U < \deg F}} \tau(R(V, U)) \sum_{\alpha \in GF(q)} \psi(\alpha) = 0. \end{aligned}$$

Therefore  $L_1(t, V)$  is a polynomial of degree  $\leq \deg F$ . Similar reasoning with  $\beta(W)$  in place of  $\alpha(W)$  shows that  $L_2(t, V)$  is a polynomial of degree  $\leq 1 + \deg F$ . In view of (2.5), (2.6a) and (2.6b), it remains to prove (2.8a) and (2.8b) for primitive  $V \neq 1$ .

To prove (2.8a), consider the double sum

$$(2.9) \quad \mu_1 := \sum_U \sum_W \psi \left( -\operatorname{Res}_\infty \frac{U(x)W(x)}{F(x)} \right) \psi(\alpha(W)) \bar{\tau}(R(V, U)),$$

where  $W = W(x)$  ranges over monic polynomials of degree  $D := \deg F$  over  $GF(q)$  and  $U = U(x)$  ranges over nonzero polynomials of degree  $< D$  over  $GF(q)$ . Write  $k = \deg U$ ,

$$(2.10) \quad W(x) = w_D x^D + w_{D-1} x^{D-1} + \cdots + w_0, \quad (w_D = 1),$$

and

$$(2.11) \quad \frac{x^k U(1/x)}{x^D F(1/x)} = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Note that  $a_0 \neq 0$  is the leading coefficient of  $U(x)$ . We have

$$(2.12) \quad \psi \left( \alpha(W) - \operatorname{Res}_\infty \frac{UW}{F} \right) = \psi \left( w_{D-1} + \sum_{i=0}^{k+1} a_{k+1-i} w_{D-i} \right).$$

For fixed  $U$ , the sum over  $W$  in (2.9) thus vanishes unless  $U(x) = -1$ . When  $U(x) = -1$ , each member of (2.12) equals  $\psi(a_1) = \psi(\alpha(F))$ . Therefore

$$(2.13) \quad \mu_1 = q^{\deg F} \tau(-1)^{\deg V} \psi(\alpha(F)).$$

On the other hand, by the proof of the last formula in [1, §2] (here primitivity is used), we have

$$(2.14) \quad \mu_1 = \varepsilon_1(V) \sigma(F) \bar{\tau}(R(V, F')) \prod_{v|F} G(-\operatorname{ord}_v V)^{\deg v}.$$

Comparison of (2.13) and (2.14) yields (2.8a).

To prove (2.8b), consider the double sum

$$(2.15) \quad \mu_2 := \sum_U \sum_Y \psi \left( -\operatorname{Res}_\infty \frac{U(x)Y(x)}{F(x)} \right) \psi(\beta(Y)) \bar{\tau}(R(V, U))$$

where  $Y$  ranges over monic polynomials of degree  $D + 1$  over  $GF(q)$  (with  $D = \deg F$ ) and  $U$  ranges over nonzero polynomials of degree  $< D$  over  $GF(q)$ . Write  $k = \deg U$  and

$$(2.16) \quad Y(x) = y_{D+1}x^{D+1} + y_Dx^D + \cdots + y_0, \quad (y_{D+1} = 1).$$

In the notation of (2.11),

$$(2.17) \quad \psi \left( \beta(Y) - \operatorname{Res}_\infty \frac{UY}{F} \right) = \psi \left( y_D^2/2 - y_{D-1} + \sum_{i=0}^{k+2} a_{k+2-i} y_{D+1-i} \right).$$

For fixed  $U$ , the sum over  $Y$  in (2.15) vanishes unless  $U(x) = 1$ . When  $U(x) = 1$ , each member of (2.17) equals  $\psi(a_2 + a_1 y_D + y_D^2/2)$  with

$$a_1 = -\alpha(F), \quad a_2 = \alpha(F)^2/2 + \beta(F).$$

Therefore

$$(2.18) \quad \mu_2 = q^D \psi(\beta(F)) \sum_{y \in GF(q)} \psi(y^2/2) = q^D \psi(\beta(F)) \phi(2) G((q-1)/2).$$

On the other hand, by the proof of the last formula in [1, §2], we have

$$(2.19) \quad \mu_2 = \varepsilon_2(V) \sigma(F) \bar{\tau}(R(V, F')) \prod_{v|F} G(-\operatorname{ord}_v V)^{\deg v}.$$

Comparison of (2.18) and (2.19) yields (2.8b).

### §3. PROOF OF THEOREMS 1.1, 1.1a, 1.1b

Let  $d$  denote the order of  $\tau^c$ . The following lemma gives useful formulas for  $P_n(a, b, c)$ ,  $P_n(a, c)$ , and  $P_n(c)$  in the case  $d | n$ . The proof of (3.1b) is elementary but for (3.1) and (3.1a) we require the Hasse-Davenport product formula [7, (7)].

LEMMA 3.1. *Let  $d$  be the smallest positive integer such that  $cd \equiv 0 \pmod{q-1}$ . If  $d | n$ , then*