

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE EVALUATION OF SELBERG CHARACTER SUMS
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Kapitel: §1. Introduction
DOI: <https://doi.org/10.5169/seals-58741>

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THE EVALUATION OF SELBERG CHARACTER SUMS

by Ronald J. EVANS

ABSTRACT. The evaluations of Selberg character sums conjectured on p. 207 of *Enseignement Math.* 27 (1981) are proved.

§1. INTRODUCTION

Many of the classical special functions over \mathbf{C} have character sum analogs over finite fields. For example, the Gauss and Jacobi sums defined in (1.1) are analogs of the gamma and beta integrals

$$\Gamma(a) = \int_0^\infty e^{-x} x^a \frac{dx}{x}, \quad \beta(a, b) = \int_0^1 x^a (1-x)^b \frac{dx}{x(1-x)}.$$

Some identities for character sums over finite fields seem more difficult to prove than their classical counterparts; compare, e.g., the Hasse-Davenport product formula for Gauss sums [7, (7)] with the Gauss multiplication formula for gamma functions. The identities for n -dimensional Selberg character sums given in Theorems 1.1, 1.1a provide further examples. Their counterparts are the well known n -dimensional Selberg integral extensions of the gamma and beta integral formulas.

The case $n = 3$ of the Selberg character sum identity in Theorem 1.1 has been used to evaluate a sum connected with the root system G_2 [8]. The case $n = 2$ is equivalent to an analog of Dixon's summation formula [11, (2.1.5)] involving hypergeometric ${}_3F_2$ character sums over finite fields. We remark that hypergeometric character sums have been used, e.g., in the computation of the number of points on hypersurfaces [13], [12], in proving congruences for Apéry numbers [14], and in graph theory [6], [9].

Let $GF(q)$ be a finite field of q elements, where q is a power of an odd prime. Fix a multiplicative character $\tau: GF(q)^* \rightarrow \mathbf{C}^*$ of order $q - 1$ and a nontrivial additive character $\psi: GF(q) \rightarrow \mathbf{C}^*$. Extend τ by defining $\tau(0) = 0$. Let $\phi = \tau^{(q-1)/2}$ be the quadratic character on $GF(q)$. For all integers a, b , define the Gauss sums $G(a)$ and Jacobi sums $J(a, b)$ by

$$(1.1) \quad G(a) = \sum_{\xi \in GF(q)^*} \tau(\xi)^a \psi(\xi), \quad J(a, b) = \sum_{1 \neq \xi \in GF(q)^*} \tau(\xi)^a \tau(1 - \xi)^b.$$

For integers $n \geq 0$ and $a, b, c > 0$, define the Selberg character sums

$$(1.2) \quad S_n(a, b, c) = \sum_E \tau((-1)^{an} E(0)^a E(1)^b \Delta_E^c) \phi(\Delta_E),$$

$$(1.2a) \quad S_n(a, c) = \sum_E \psi(e_{n-1}) \tau(E(0)^a \Delta_E^c) \phi(\Delta_E),$$

$$(1.2b) \quad S_n(c) = \sum_E \psi(e_{n-1}^2/2 - e_{n-2}) \tau(\Delta_E)^c \phi(\Delta_E),$$

where each sum is over all monic polynomials

$$(1.3) \quad E = E(x) = x^n + e_{n-1}x^{n-1} + e_{n-2}x^{n-2} + \cdots + e_0$$

of degree n over $GF(q)$, and where Δ_E denotes the discriminant of E (with the convention that $\Delta_E = 1$ when $\deg(E) \leq 1$). Define the following products:

$$(1.4) \quad P_n(a, b, c) = \prod_{j=0}^{n-1} \frac{G(a+jc)G(b+jc)G(c+jc)\bar{G}(a+b+(n-1+j)c)}{qG(c)},$$

$$(1.4a) \quad P_n(a, c) = \prod_{j=0}^{n-1} \frac{G(a+jc)G(c+jc)}{G(c)},$$

$$(1.4b) \quad P_n(c) = \prod_{j=0}^{n-1} \frac{G(c+jc)\phi(2)G((q-1)/2)}{G(c)},$$

where \bar{G} denotes the complex conjugate of G .

The object of this paper is to prove Theorems 1.1, 1.1a, and 1.1b below. These results, analogs of n -dimensional integral formulas of Selberg [3, (1.1), (1.3), (1.2)], [2], verify conjectures made in 1981 [7, (29), (29a), (29b)]. The decisive breakthrough came in 1990 when Anderson [1] proved a somewhat weakened form of Theorem 1.1. The proofs here are based on modifications of the method in [1]. The modifications are designed to handle complications arising from "imprimitive" L -functions (see §2).

THEOREM 1.1. *For all integers $n, a, b, c > 0$, if none of*

$$a + b + (n-1+j)c \quad (0 \leq j \leq n-1)$$

are divisible by $q-1$, then $S_n(a, b, c) = P_n(a, b, c)$.

THEOREM 1.1a. For all integers $n, a, c > 0$, $S_n(a, c) = P_n(a, c)$.

THEOREM 1.1b. For all integers $n, c > 0$, $S_n(c) = P_n(c)$.

Given a monic polynomial E over $GF(q)$, define $\sigma(E) = 0$ if E is not squarefree, $\sigma(E) = 1$ if $E = 1$, and otherwise let $\sigma(E)$ denote the sign of the permutation of the zeros of E effected by the q^{th} power automorphism of $\overline{GF(q)}$. For odd q , $\sigma(E) = \phi(\Delta_E)$. If $\phi(\Delta_E)$ is replaced by $\sigma(E)$ in the definitions (1.2), (1.2a) of $S_n(a, b, c)$, $S_n(a, c)$, then Theorems 1.1 and 1.1a remain valid without the stipulation “ q odd”; the proofs for even q are virtually the same. This observation is due to Serre; see [1].

The following result is equivalent to Theorem 1.1, as was shown in [10, p. 116].

THEOREM 1.2. For integers $n, a, b, c > 0$, if none of $a + jc$ ($0 \leq j \leq n-1$) are divisible by $q-1$, or if none of $b + jc$ ($0 \leq j \leq n-1$) are divisible by $q-1$, or if none of $a + b + (n-1+j)c$ ($0 \leq j \leq n-1$) are divisible by $q-1$, then $S_n(a, b, c) = P_n(a, b, c)$.

Theorems 1.3 and 1.4 below, analogs of more recent Selberg integral formulas (see [4]), were stated as conjectures in [5]. They are consequences of Theorems 1.1a and 1.1b, respectively, as is shown in [5, Theorems 2.2 and 2.5].

THEOREM 1.3. For all integers $n, a, b, c > 0$,

$$\sum_E \tau(E(0)^a (1 + e_{n-1})^b \Delta_E^c) \phi(\Delta_E) \\ = \begin{cases} \frac{G(-b - na - n(n-1)c)}{G(-b)} P_n(a, c), & \text{if } b \not\equiv 0 \pmod{q-1} \\ \frac{\tau(-1)^{an} G(b)}{G(b + na + n(n-1)c)} P_n(a, c), & \text{if } b + na + n(n-1)c \\ & \not\equiv 0 \pmod{q-1}, \end{cases}$$

where the sum is over all polynomials E of degree n given by (1.3).

THEOREM 1.4. For $w \in GF(q)^*$ and all integers $n, b, c > 0$ with $b \not\equiv 0 \pmod{q-1}$,

$$\sum_E \tau((w + e_{n-1}^2/2 - e_{n-2})^b \Delta_E^c) \phi(\Delta_E) \\ = \tau(w)^{b+n(q-1)/2+cn(n-1)/2} \frac{G(-b - cn(n-1)/2 - n(q-1)/2)}{G(-b)} P_n(c),$$

where the sum is over all polynomials E of degree n given by (1.3).

Acknowledgement. We are very grateful to G. W. Anderson for helpful correspondence on L -functions and character sums.

§2. L -FUNCTIONS

Throughout this section, V denotes a *monic* polynomial over $GF(q)$, and v ranges over the distinct monic irreducible factors of V over $GF(q)$. Write

$$(2.1) \quad V = \prod_{v|V} v^{\text{ord}_v V}, \quad F = F_V = \prod_{v|V} v.$$

If no exponent $\text{ord}_v V$ in (2.1) is divisible by $q - 1$, then V is said to be *primitive*. Note that $V = 1$ is primitive. For any monic polynomial

$$(2.2) \quad W = W(x) = x^n + w_{n-1}x^{n-1} + w_{n-2}x^{n-2} + \cdots + w_0$$

over $GF(q)$, set

$$(2.3) \quad \alpha(W) = w_{n-1}, \quad \beta(W) = w_{n-1}^2/2 - w_{n-2}.$$

Define the L -functions

$$(2.4) \quad L(t, V) = \sum_W \tau(R(V, W)) t^{\deg W},$$

$$(2.4a) \quad L_1(t, V) = \sum_W \psi(\alpha(W)) \tau(R(V, W)) t^{\deg W},$$

$$(2.4b) \quad L_2(t, V) = \sum_W \psi(\beta(W)) \tau(R(V, W)) t^{\deg W},$$

where in each sum, W ranges over all monic polynomials over $GF(q)$, and $R(V, W)$ is the resultant of V and W . It is easily checked that

$$(2.5) \quad \begin{aligned} L(t, 1) &= (1 - qt)^{-1}, & L_1(t, 1) &= 1, \\ L_2(t, 1) &= 1 + \phi(2)G((q-1)/2)t. \end{aligned}$$

Since the summands in (2.4), (2.4a), (2.4b) are multiplicative in W , each of the L -functions has an Euler product expansion. Thus we have the following result.

LEMMA 2.1. *Write $V = GH$ where G and H are monic, relatively prime polynomials over $GF(q)$ with G primitive and H a $(q-1)$ th power. Then*

$$(2.6) \quad L(t, V) = L(t, G) \prod_{v|H} (1 - \tau(R(G, v)) t^{\deg v}),$$

$$(2.6a) \quad L_1(t, V) = L_1(t, G) \prod_{v|H} (1 - \psi(\alpha(v)) \tau(R(G, v)) t^{\deg v}),$$