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We have, for  $1 \leq i \leq N$  and  $1 \leq j \leq N$ ,

$$(2.3) \quad x \equiv s_{i,j} q^j - d - 1(p^i q^j) \quad \text{where} \quad \begin{cases} s_{1,1} = 1 \\ 1 \leq s_{i,j} \leq p^i - 1 \end{cases},$$

whence

$$(2.4) \quad \begin{aligned} H(k, x) &\geq \frac{1}{2} + \sum_{i=1}^N \frac{(1-p)}{p^i} \left( -\frac{1}{2} + \frac{1}{p^i} \right) + \sum_{j=1}^N \frac{(1-q)}{q^j} \left( -\frac{1}{2} + \frac{d+1}{q^j} \right) \\ &+ \frac{(p-1)(q-1)}{pq} \left( \frac{1}{2} - \frac{q-d-1}{pq} \right) \\ &+ \sum_{\substack{1 \leq i, j \leq N \\ (i, j) \neq (1, 1)}} \frac{(p-1)(q-1)}{p^i q^j} \left( \frac{1}{2} - \frac{(p^i-1)q^j-d-1}{p^i q^j} \right) + o_N(1). \end{aligned}$$

The right side of (2.4) tends to the right side of (2.1) as  $N \rightarrow \infty$ , and the theorem is proved in virtue of (0.15).  $\square$

### PROOF OF THEOREM 3

The function  $f_r$  defined in (0.11) satisfies, provided  $r \geq 3$ ,

$$(3.1) \quad f_r(p_2, \dots, p_r) < f_{r-1}(p_2, \dots, p_{r-1}) \leq p_2,$$

and thus the condition

$$(3.2) \quad f_r(p_2, \dots, p_r) \geq x$$

implies, for any  $x$ , that

$$(3.3) \quad p_2 \begin{cases} > x & \text{if } r \geq 3, \\ \geq x & \text{if } r = 2. \end{cases}$$

Also note that, since

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{\gamma_k(n)}{n} = \prod_{p|k} \left( 1 + (1-p) \sum_{i \geq 1} \frac{1}{p^i} \right) = 0,$$

we have in fact

$$(3.5) \quad H(k, x) = - \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left\{ \frac{x}{n} \right\}.$$

After these preliminaries let  $N \geq 1$  be as in (0.11), and define

$$(3.6) \quad x = x_N := p^N - 1,$$

where we denote  $p_1$  simply by  $p$ . For  $r \geq 3$  (3.2) and (3.3) imply that

$$(3.7) \quad p_2 > x,$$

and (3.7) clearly remains true for  $r = 2$  if  $p \neq 2$ . Hence

$$\begin{aligned} H(k, x) &= (p - 1) \left( \frac{p - 1}{p^2} + \frac{p^2 - 1}{p^4} + \dots + \frac{p^{N-1} - 1}{p^{2N-2}} \right) - (p^N - 1) \sum_{n \geq N} \frac{\gamma_k(n)}{n^2} \\ &= - \frac{(p^N - 1)(p^{N-1} - 1)}{p^{N-1}(p + 1)} - (p^N - 1) \sum_{n \geq 2} \frac{\gamma_k(n)}{n^2} \\ (3.8) \quad &= (p^N - 1) \left( \frac{(1 - p^{N-1})}{(p + 1)p^{N-1}} - \prod_{p|k} \left( 1 + (1 - p) \sum_{i \geq 1} \frac{1}{p^{2i}} \right) + 1 \right) \\ &= (p^N - 1) \left( \frac{p^N + 1}{(p + 1)p^{N-1}} - \frac{k}{\sigma(k)} \right). \end{aligned}$$

Now when a rational number  $P/Q$  is less than an integer  $M$ , we may conclude that  $M - P/Q \geq 1/Q$ . Thus from (0.11) we have

$$(3.9) \quad \frac{p}{p + 1} \cdot \frac{\sigma(k)}{k} = \frac{\sigma(k/p)}{k/p} \geq \frac{p^{N+1} - (\sigma(k/p) - k/p)^{-1}}{p^{N+1} - 1 - (\sigma(k/p) - k/p)^{-1}},$$

whence from (3.8)

$$(3.10) \quad H(k, x) \geq \frac{k}{\sigma(k)} + \frac{1}{(p + 1)} \left( 1 + \frac{1}{(\sigma(k/p) - k/p)p^{N+1} - 1} - \frac{1}{p^{N-1}} \right).$$

On appealing to Lemma 0 this concludes the proof of the theorem.  $\square$

*Last Remark.* Neither of the estimates (2.1) (of Theorem 2') and (0.12) (of Theorem 3) is better than the other in all cases considered by both theorems. For instance in the case where  $k = pq = p(p + d)$  with  $p$  and  $q$  odd primes and  $2 \leq d \leq p - 2$ , there is some positive number  $\varepsilon$  depending on  $p$ , satisfying

$$(3.11) \quad \frac{13/4}{\sqrt{p}} < \varepsilon < \frac{8.06}{\sqrt{p}},$$

and such that (2.1) is better than (the first estimate of) (0.12) if  $d < 2\sqrt{p} + 2 + \varepsilon$ , and is not as good if  $d > 2\sqrt{p} + 2 + \varepsilon$ .

ADDED IN PROOF. Recently, S. D. Adhikari and K. Soundararajan gave a much simpler proof of (0.9) than mine in “Towards the exact nature of a certain error term, II” (preprint).

## REFERENCES

- [A] ADHIKARI, S. D. Towards the exact nature of a certain error term. To appear in *Arch. Math.* 11.
- [AB] ADHIKARI, S. D. and R. BALASUBRAMANIAN. A note on a certain error term. *Arch. Math.* 56 (1991), 37-40.
- [ABS] ADHIKARI, S. D., R. BALASUBRAMANIAN and A. SANKARANARAYANAN. On an error term related to the greatest divisor of  $n$ , which is prime to  $k$ . *Indian J. Pure Appl. Math.* 19 (1988), 830-841.
- [HM] HERZOG, J. and T. MAXSEIN. On the behaviour of a certain error term. *Arch. Math.* 50 (1988), 145-155.
- [JV] JOSHI, V. S. and A. M. VAIDYA. Average behaviour of the largest  $k$ -prime divisor of an integer. *Coll. Math. Soc. János Bolyai* 34; *Topics in Classical Number Theory* (1981), 791-806.
- [P1] PÉTERMANN, Y.-F. S. Distribution of values of a real function. Means, moments, and symmetry. *Manuscripta Math.* 69 (1990), 305-318.
- [P2] —— Oscillations d'un terme d'erreur lié à la fonction totient de Jordan. A paraître: *Sém. Théorie des Nombres de Bordeaux*.
- [Su] SURYANARAYANA, D. The greatest divisor of  $n$  which is prime to  $k$ . *Math. Stud.* 37 (1969), 145-157.
- [W] WINTNER, A. On the asymptotic distribution of the remainder term of the prime-number theorem. *Amer. J. Math.* 57 (1935), 534-538.

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