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Artikel: ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k

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Kapitel: Proof of Theorem 3

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We have, for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant N$,

$$x \equiv s_{i,j} q^j - d - 1 (p^i q^j) \quad \text{where} \quad \begin{cases} s_{1,1} = 1 \\ 1 \leqslant s_{i,j} \leqslant p^i - 1 , \end{cases} \tag{2.3}$$

whence

$$\begin{aligned} H(k,x) \geqslant \frac{1}{2} &+ \sum_{i=1}^{N} \frac{(1-p)}{p^i} \left(-\frac{1}{2} + \frac{1}{p^i} \right) + \sum_{j=1}^{N} \frac{(1-q)}{q^j} \left(-\frac{1}{2} + \frac{d+1}{q^j} \right) \\ &+ \frac{(p-1)(q-1)}{pq} \left(\frac{1}{2} - \frac{q-d-1}{pq} \right) \\ &+ \sum_{\substack{1 \leqslant i,j \leqslant N \\ (i,j) \neq (1,1)}} \frac{(p-1)(q-1)}{p^i q^j} \left(\frac{1}{2} - \frac{(p^i - 1) q^j - d - 1}{p^i q^j} \right) + o_N(1) . \end{aligned} \tag{2.4}$$

The right side of (2.4) tends to the right side of (2.1) as $N \to \infty$, and the theorem is proved in virtue of (0.15). □

Proof of Theorem 3

The function f_r defined in (0.11) satisfies, provided $r \geqslant 3$,

$$f_r(p_2, \ldots, p_r) < f_{r-1}(p_2, \ldots, p_{r-1}) \leqslant p_2 , \tag{3.1}$$

and thus the condition

$$f_r(p_2, \ldots, p_r) \geqslant x \tag{3.2}$$

implies, for any x, that

$$p_2 \begin{cases} > x & \text{if} \quad r \geqslant 3 , \\ \geqslant x & \text{if} \quad r = 2 . \end{cases} \tag{3.3}$$

Also note that, since

$$\sum_{n=1}^{\infty} \frac{\gamma_k(n)}{n} = \prod_{p \mid k} \left(1 + (1-p) \sum_{i \geqslant 1} \frac{1}{p^i} \right) = 0 , \tag{3.4}$$

we have in fact

$$H(k,x) = - \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left\{ \frac{x}{n} \right\} . \tag{3.5}$$

After these preliminaries let $N \geqslant 1$ be as in (0.11), and define

(3.6) $$x = x_N := p^N - 1\ ,$$

where we denote p_1 simply by p. For $r \geqslant 3$ (3.2) and (3.3) imply that

(3.7) $$p_2 > x\ ,$$

and (3.7) clearly remains true for $r = 2$ if $p \neq 2$. Hence

(3.8)
$$\begin{aligned} H(k,x) &= (p-1)\left(\frac{p-1}{p^2} + \frac{p^2-1}{p^4} + \dots + \frac{p^{N-1}-1}{p^{2N-2}}\right) - (p^N - 1 \sum_{n \geqslant N} \frac{\gamma_k(n)}{n^2} \\ &= -\frac{(p^N-1)(p^{N-1}-1)}{p^{N-1}(p+1)} - (p^N-1) \sum_{n \geqslant 2} \frac{\gamma_k(n)}{n^2} \\ &= (p^N-1)\left(\frac{(1-p^{N-1})}{(p+1)p^{N-1}} - \prod_{p \mid k}\left(1 + (1-p)\sum_{i \geqslant 1}\frac{1}{p^{2i}}\right) + 1\right) \\ &= (p^N-1)\left(\frac{p^N+1}{(p+1)p^{N-1}} - \frac{k}{\sigma(k)}\right). \end{aligned}$$

Now when a rational number P/Q is less than an integer M, we may conclude that $M - P/Q \geqslant 1/Q$. Thus from (0.11) we have

(3.9) $$\frac{p}{p+1} \cdot \frac{\sigma(k)}{k} = \frac{\sigma(k/p)}{k/p} \geqslant \frac{p^{N+1} - (\sigma(k/p) - k/p)^{-1}}{p^{N+1} - 1 - (\sigma(k/p) - k/p)^{-1}}\ ,$$

whence from (3.8)

(3.10) $$H(k,x) \geqslant \frac{k}{\sigma(k)} + \frac{1}{(p+1)}\left(1 + \frac{1}{(\sigma(k/p) - k/p)p^{N+1} - 1} - \frac{1}{p^{N-1}}\right).$$

On appealing to Lemma 0 this concludes the proof of the theorem. □

Last Remark. Neither of the estimates (2.1) (of Theorem 2′) and (0.12) (of Theorem 3) is better than the other in all cases considered by both theorems. For instance in the case where $k = pq = p(p+d)$ with p and q odd primes and $2 \leqslant d \leqslant p-2$, there is some positive number ε depending on p, satisfying

(3.11) $$\frac{13/4}{\sqrt{p}} < \varepsilon < \frac{8.06}{\sqrt{p}}\ ,$$

and such that (2.1) is better than (the first estimate of) (0.12) if $d < 2\sqrt{p} + 2 + \varepsilon$, and is not as good if $d > 2\sqrt{p} + 2 + \varepsilon$.

ADDED IN PROOF. Recently, S. D. Adhikari and K. Soundararajan gave a much simpler proof of (0.9) than mine in "Towards the exact nature of a certain error term, II" (preprint).

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