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$$(1.16) \quad D_h(\delta) = D_{0,h}(\delta) \quad (\delta = \beta \text{ or } \gamma) .$$

In view of (0.14) this implies that

$$\begin{aligned} 1 - D_h(S_k - \varepsilon) &\geq D_h\left(S_k - \frac{3\varepsilon}{4}\right) - D_h(S_k - \varepsilon) \\ (1.17) \quad &\geq D_h(\gamma) - D_h(\beta) = D_{0,h}(\gamma) - D_{0,h}(\beta) \geq \frac{1}{L} \cdot \frac{\varepsilon}{5} \cdot \frac{\sigma(k)}{k} . \quad \square \end{aligned}$$

*Remark.* I studied in [P2] an error term associated with the  $k$ -th Jordan totient function (for  $k \geq 2$ ), that can be expressed in terms of the function

$$(1.18) \quad g_k(x) := - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \psi\left(\frac{x}{n}\right) ,$$

where  $\mu$  denotes the Moebius function, and I proved by a direct method that

$$(1.19) \quad \liminf_{x \rightarrow \infty} g_k(x) = - \limsup_{x \rightarrow \infty} g_k(x) .$$

This can also be obtained by an argument similar to the above proof.

## 2. THE CASE $\omega(k) = 2$

In this section we obtain an estimate more general than (0.10) of Theorem 2.

**THEOREM 2'.** *Let  $k = pq$  where  $p < q$  and  $p$  and  $q$  are prime numbers, and let  $d = q - ps$  with  $1 \leq d \leq p - 1$  be the remainder of the Euclidean division of  $q$  by  $p$ . Then we have*

$$(2.1) \quad S_k \geq \frac{k}{\sigma(k)} + \frac{1}{(p+1)} - \frac{pd}{(p+1)(q+1)} + \frac{(p+1)(p-2)(q-1)}{p^2q} .$$

The right side of (2.1) is easily seen to exceed  $k/\sigma(k)$  for any  $p$  and  $q$ . And in the special case where  $p = 2$  it reduces to  $\left(q - \frac{1}{3}\right)/(q+1)$ .

*Proof.* Let  $N$  be a positive integer. We define, modulo  $p^N q^N$ , the integer  $x = x_N$  by the system of congruences

$$(2.2) \quad \begin{cases} x \equiv -1(p^N) \\ x \equiv -d - 1(q^N) . \end{cases}$$

We have, for  $1 \leq i \leq N$  and  $1 \leq j \leq N$ ,

$$(2.3) \quad x \equiv s_{i,j} q^j - d - 1(p^i q^j) \quad \text{where} \quad \begin{cases} s_{1,1} = 1 \\ 1 \leq s_{i,j} \leq p^i - 1 \end{cases},$$

whence

$$(2.4) \quad \begin{aligned} H(k, x) &\geq \frac{1}{2} + \sum_{i=1}^N \frac{(1-p)}{p^i} \left( -\frac{1}{2} + \frac{1}{p^i} \right) + \sum_{j=1}^N \frac{(1-q)}{q^j} \left( -\frac{1}{2} + \frac{d+1}{q^j} \right) \\ &+ \frac{(p-1)(q-1)}{pq} \left( \frac{1}{2} - \frac{q-d-1}{pq} \right) \\ &+ \sum_{\substack{1 \leq i, j \leq N \\ (i, j) \neq (1, 1)}} \frac{(p-1)(q-1)}{p^i q^j} \left( \frac{1}{2} - \frac{(p^i-1)q^j-d-1}{p^i q^j} \right) + o_N(1). \end{aligned}$$

The right side of (2.4) tends to the right side of (2.1) as  $N \rightarrow \infty$ , and the theorem is proved in virtue of (0.15).  $\square$

### PROOF OF THEOREM 3

The function  $f_r$  defined in (0.11) satisfies, provided  $r \geq 3$ ,

$$(3.1) \quad f_r(p_2, \dots, p_r) < f_{r-1}(p_2, \dots, p_{r-1}) \leq p_2,$$

and thus the condition

$$(3.2) \quad f_r(p_2, \dots, p_r) \geq x$$

implies, for any  $x$ , that

$$(3.3) \quad p_2 \begin{cases} > x & \text{if } r \geq 3, \\ \geq x & \text{if } r = 2. \end{cases}$$

Also note that, since

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{\gamma_k(n)}{n} = \prod_{p|k} \left( 1 + (1-p) \sum_{i \geq 1} \frac{1}{p^i} \right) = 0,$$

we have in fact

$$(3.5) \quad H(k, x) = - \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left\{ \frac{x}{n} \right\}.$$