$$D_h(\delta) = D_{0,h}(\delta) \quad (\delta = \beta \text{ or } \gamma)\ . \tag{1.16}$$

In view of (0.14) this implies that

$$\begin{aligned}1 - D_h(S_k - \varepsilon) &\geqslant D_h\left(S_k - \frac{3\varepsilon}{4}\right) - D_h(S_k - \varepsilon)\\ &\geqslant D_h(\gamma) - D_h(\beta) = D_{0,h}(\gamma) - D_{0,h}(\beta) \geqslant \frac{1}{L}\cdot\frac{\varepsilon}{5}\cdot\frac{\sigma(k)}{k}\ . \quad \square\end{aligned} \tag{1.17}$$

Remark. I studied in [P2] an error term associated with the k-th Jordan totient function (for $k \geqslant 2$), that can be expressed in terms of the function

$$g_k(x) := -\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}\,\psi\left(\frac{x}{n}\right), \tag{1.18}$$

where μ denotes the Moebius function, and I proved by a direct method that

$$\liminf_{x\to\infty} g_k(x) = -\limsup_{x\to\infty} g_k(x)\ . \tag{1.19}$$

This can also be obtained by an argument similar to the above proof.

2. The case $\omega(k) = 2$

In this section we obtain an estimate more general than (0.10) of Theorem 2.

Theorem 2′. *Let $k = pq$ where $p < q$ and p and q are prime numbers, and let $d = q - ps$ with $1 \leqslant d \leqslant p - 1$ be the remainder of the Euclidean division of q by p. Then we have*

$$S_k \geqslant \frac{k}{\sigma(k)} + \frac{1}{(p+1)} - \frac{pd}{(p+1)(q+1)} + \frac{(p+1)(p-2)(q-1)}{p^2 q}\ . \tag{2.1}$$

The right side of (2.1) is easily seen to exceed $k/\sigma(k)$ for any p and q. And in the special case where $p = 2$ it reduces to $\left(q - \frac{1}{3}\right)/(q+1)$.

Proof. Let N be a positive integer. We define, modulo $p^N q^N$, the integer $x = x_N$ by the system of congruences

$$\begin{cases} x \equiv -1(p^N) \\ x \equiv -d-1(q^N)\ . \end{cases} \tag{2.2}$$

We have, for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant N$,

$$x \equiv s_{i,j} q^j - d - 1 (p^i q^j) \quad \text{where} \quad \begin{cases} s_{1,1} = 1 \\ 1 \leqslant s_{i,j} \leqslant p^i - 1 , \end{cases} \tag{2.3}$$

whence

$$\begin{aligned} H(k,x) \geqslant \frac{1}{2} + \sum_{i=1}^{N} \frac{(1-p)}{p^i} \left(-\frac{1}{2} + \frac{1}{p^i} \right) + \sum_{j=1}^{N} \frac{(1-q)}{q^j} \left(-\frac{1}{2} + \frac{d+1}{q^j} \right) \\ + \frac{(p-1)(q-1)}{pq} \left(\frac{1}{2} - \frac{q-d-1}{pq} \right) \\ + \sum_{\substack{1 \leqslant i,j \leqslant N \\ (i,j) \neq (1,1)}} \frac{(p-1)(q-1)}{p^i q^j} \left(\frac{1}{2} - \frac{(p^i - 1) q^j - d - 1}{p^i q^j} \right) + o_N(1) . \end{aligned} \tag{2.4}$$

The right side of (2.4) tends to the right side of (2.1) as $N \to \infty$, and the theorem is proved in virtue of (0.15). □

Proof of Theorem 3

The function f_r defined in (0.11) satisfies, provided $r \geqslant 3$,

$$f_r(p_2, \ldots, p_r) < f_{r-1}(p_2, \ldots, p_{r-1}) \leqslant p_2 , \tag{3.1}$$

and thus the condition

$$f_r(p_2, \ldots, p_r) \geqslant x \tag{3.2}$$

implies, for any x, that

$$p_2 \begin{cases} > x & \text{if} \quad r \geqslant 3 , \\ \geqslant x & \text{if} \quad r = 2 . \end{cases} \tag{3.3}$$

Also note that, since

$$\sum_{n=1}^{\infty} \frac{\gamma_k(n)}{n} = \prod_{p \mid k} \left(1 + (1-p) \sum_{i \geqslant 1} \frac{1}{p^i} \right) = 0 , \tag{3.4}$$

we have in fact

$$H(k,x) = - \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left\{ \frac{x}{n} \right\} . \tag{3.5}$$