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Artikel: ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k

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Kapitel: 1. Proof of Theorem 1

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$$H(k,x) := \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

(0.14) $$H(k,x) = H(k,[x]) - \frac{k}{\sigma(k)} \{x\} + o(1) ,$$

and

LEMMA 0. *We have*

(0.15) $$S_k = \sup_{n \in \mathbf{Z}} H(k,n).$$

Proof. In view of (0.13), (0.14), and the definition of $H(k,x)$, it is sufficient to show that

(0.16) $$\limsup_{N \to \infty, N \in \mathbf{N}} H(k,N) = \sup_{n \in \mathbf{Z}} H(k,n) .$$

When $k = 1$ this is easily verified; when $k \geqslant 2$ and $N \in \mathbf{Z}$ we define for each positive integer i the positive integer $N_i := (|N| + 1)k^i + N$ and we see, since

(0.17) $$\sum_{m \nmid k^i} \frac{\gamma_k(m)}{m} \to 0 \ (i \to \infty) ,$$

and since for every divisor m of k^i we have $\{N_i/m\} = \{N/m\}$, that

(0.18) $$\lim_{i \to \infty} H(k,N_i) = H(k,N) . \quad \square$$

1. PROOF OF THEOREM 1

We first set some terminology. Let $g\colon [1, \infty] \to \mathbf{R}$ be a measurable function, and consider as in [P1]

(1.1) $$D_0(u) = D_{0,g}(u) := \lim_{x \to \infty} \frac{1}{x} \mu \{t \in [0,x], g(t) \leqslant x\} ,$$

and

(1.2) $$D_0(u^+) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) , \quad D_0(u^-) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) ,$$

where μ denotes the Lebesgue measure and E the set of values for which D_0 exists. In case D_0 exists almost everywhere we say, following A. Wintner [W, p. 537], that g *possesses an asymptotic distribution function.* If (and only if) this is so we define an associated function $D = D_g \colon \mathbf{R} \to [0, 1]$ by

$$D(u) := \frac{1}{2}\left(D_0(u^+) + D_0(u^-)\right). \tag{1.3}$$

And it is this function D we call *the asymptotic distribution function of* g. The convention is of course abusive[2]; we point out however that D_0 exists and coincides with D at least wherever D is continuous (which, since D is a distribution function, is the case almost everywhere).

The first two statements of Theorem 1, $D = D_h$ exists and is continuous, are proved through a straightforward application of two theorems from [P1].

Indeed, it is easy to see that

$$\sum_{n \leqslant x} \gamma_k(n) = O\left((\log x)^{\omega(k)}\right) = 0 \cdot x + o(x) \tag{1.4}$$

holds, and that for any function $z = z(x) \to \infty$ $(x \to \infty)$ (and in particular for a slowly increasing function), we have

$$H(k, x) = \sum_{n \leqslant z} \frac{\gamma_k(n)}{n}\left(-\psi\left(\frac{x}{n}\right)\right) + o(1), \tag{1.5}$$

where $\psi(y)$ denotes the function $\{y\} - \frac{1}{2}$ which satisfies

$$\int_0^1 \psi(t)\,dt = 0. \tag{1.6}$$

In the notation of [P1] the properties (1.4) through (1.6) are expressed by writing $h \in C_z(\psi_k, -\psi)$. Thus from Theorem 4 of that paper we have the existence of D_h. And since ψ is odd almost everywhere Theorem 5 of [P1] tells us that D_h is symmetric.

We pass now to the third assertion of the theorem, namely that $I_k = -S_k$. We denote by S the bounded support of D_h and by $-s$ and s its

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[2]) Its purpose is to ensure that D be *normalized,* i.e. that the relation

$$D(u) = \frac{1}{2}\left(D(u^+) + D(u^-)\right)$$

hold for every real number u.

greatest lower bound and least upper bound: we have $I_k \leqslant -s < s \leqslant S_k$. We show that

$$(1.7) \qquad I_k = -s = -S_k$$

holds by ensuring that

$$(1.8) \qquad 0 < D_h(\alpha) < 1 \quad \text{for every} \quad \alpha \in (I_k, S_k)\,.$$

We prove here that $D_h(S_k - \varepsilon) < 1$ for every $\varepsilon > 0$; the rest of the proof is similar. There is an increasing sequence of natural numbers n_i with $H(k, n_i) \to S_k$ $(i \to \infty)$, and thus we may select some natural number N satisfying

$$(1.9) \qquad H(k, N) > S_k - \frac{\varepsilon}{4}$$

and

$$(1.10) \qquad \frac{1}{2} \sum_{n > N} \frac{|\gamma_k(n)|}{n} < \frac{\varepsilon}{4}\,.$$

Hence if we define

$$(1.11) \qquad H^*(k, N, M) := \sum_{n \leqslant N} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{M}{n} \right\} \right).$$

we have

$$(1.12) \qquad H^*(k, N, N) > S_k - \frac{\varepsilon}{2}\,.$$

Also, if L is the least common multiple of the integers $1, 2, \ldots, N$, then

$$(1.13) \qquad H^*(k, N, mL + N) = H^*(k, N, N)$$

for every integer m, and it follows from (1.12) and (1.10) that

$$(1.14) \qquad H(k, mL + N) > S_k - \frac{3\varepsilon}{4}$$

for every integer m. Now since $D_{0,h}$ exists and coincides with D_h almost everywhere we can find two numbers β and γ satisfying

$$(1.15) \qquad S_k - \varepsilon \leqslant \beta < \beta + \frac{\varepsilon}{5} \leqslant \gamma \leqslant S_k - \frac{3\varepsilon}{4}$$

and

$$D_h(\delta) = D_{0,h}(\delta) \quad (\delta = \beta \text{ or } \gamma)\,. \tag{1.16}$$

In view of (0.14) this implies that

$$\begin{aligned} 1 - D_h(S_k - \varepsilon) &\geqslant D_h\left(S_k - \frac{3\varepsilon}{4}\right) - D_h(S_k - \varepsilon) \\ &\geqslant D_h(\gamma) - D_h(\beta) = D_{0,h}(\gamma) - D_{0,h}(\beta) \geqslant \frac{1}{L} \cdot \frac{\varepsilon}{5} \cdot \frac{\sigma(k)}{k}\,. \quad \square \end{aligned} \tag{1.17}$$

Remark. I studied in [P2] an error term associated with the k-th Jordan totient function (for $k \geqslant 2$), that can be expressed in terms of the function

$$g_k(x) := -\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}\, \psi\left(\frac{x}{n}\right), \tag{1.18}$$

where μ denotes the Moebius function, and I proved by a direct method that

$$\liminf_{x\to\infty} g_k(x) = -\limsup_{x\to\infty} g_k(x)\,. \tag{1.19}$$

This can also be obtained by an argument similar to the above proof.

2. The case $\omega(k) = 2$

In this section we obtain an estimate more general than (0.10) of Theorem 2.

THEOREM 2′. *Let $k = pq$ where $p < q$ and p and q are prime numbers, and let $d = q - ps$ with $1 \leqslant d \leqslant p - 1$ be the remainder of the Euclidean division of q by p. Then we have*

$$S_k \geqslant \frac{k}{\sigma(k)} + \frac{1}{(p+1)} - \frac{pd}{(p+1)(q+1)} + \frac{(p+1)(p-2)(q-1)}{p^2 q}\,. \tag{2.1}$$

The right side of (2.1) is easily seen to exceed $k/\sigma(k)$ for any p and q. And in the special case where $p = 2$ it reduces to $\left(q - \frac{1}{3}\right)/(q+1)$.

Proof. Let N be a positive integer. We define, modulo $p^N q^N$, the integer $x = x_N$ by the system of congruences

$$\begin{cases} x \equiv -1(p^N) \\ x \equiv -d - 1(q^N)\,. \end{cases} \tag{2.2}$$