

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	37 (1991)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k
Autor:	PÉTERMANN, Y.-F. S.
Kapitel:	0. Introduction and statement of the results
DOI:	https://doi.org/10.5169/seals-58739

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR
OF n PRIME TO A FIXED INTEGER k

by Y.-F. S. PETERMANN

RÉSUMÉ. On étudie le comportement de la fonction bornée $h_k(x) := x^{-1}E_k(x)$, où $E_k(x) := \sum_{n \leq x} \delta_k(n) - (k/2\sigma(k))x^2$ est le terme irrégulier du comportement asymptotique moyen de $\delta_k(n)$, le plus grand diviseur de n premier à k (et où l'on peut sans perte supposer que k est sans facteur carré). On s'intéresse plus particulièrement aux nombres I_k et S_k , les \liminf et \limsup de $h_k(x)$ (lorsque $x \rightarrow \infty$), dont les valeurs exactes ne sont connues que si $k = 1$ ou si k est un nombre premier (Joshi et Vaidya [JV]). En établissant l'existence et la symétrie de la fonction de répartition de $h_k(n)$ (au sens de Wintner), on simplifie le problème en démontrant que $I_k = -S_k$. Puis, pour tous les k non premiers et sans facteur carré, on améliore explicitement l'estimation $S_k \geq k/\sigma(k)$ (de Herzog et Maxsein [HM], et indépendamment Adhikari, Balasubramanian et Sankaranarayanan [ABS]).

0. INTRODUCTION AND STATEMENT OF THE RESULTS

For a fixed natural number k we denote by $\delta_k(n)$ the largest divisor of n which is prime to k . If κ is the squarefree core of k we have $\delta_k(n) = \delta_\kappa(n)$, and we shall assume from now on that k is squarefree. We define the associated error term

$$(0.1) \quad E_k(x) := \sum_{n \leq x} \delta_k(n) - \frac{k}{2\sigma(k)} x^2,$$

where σ is the sum-of-divisors function. The behaviour of $E_k(x)$ has been investigated in [Su], [JV], [HM], [ABS], [AB], and very recently in [A]. It is known that [JV]

$$(0.2) \quad E_k(x) = O(x)$$

and that [JV, HM, ABS]¹⁾

$$(0.3) \quad E_k(x) = \Omega_{\pm}(x) .$$

However, the exact values of the \limsup and \liminf of $E_k(x)/x$ are not known, except in the special case where k is a prime p (and of course when $k = 1$); we have [JV]

$$(0.4) \quad \limsup_{x \rightarrow \infty} \frac{E_p(x)}{x} = \frac{p}{p+1} \quad \text{and} \quad \liminf_{x \rightarrow \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1} .$$

Let us from now on use the notation

$$(0.5) \quad S_k := \limsup_{x \rightarrow \infty} \frac{E_k(x)}{x} \quad \text{and} \quad I_k := \liminf_{x \rightarrow \infty} \frac{E_k(x)}{x} .$$

When the number $\omega(k)$ of (distinct) prime divisors of k exceeds 1, the best estimates known so far are on the one hand [HM, ABS]

$$(0.6) \quad S_k \geq \frac{k}{\sigma(k)} \quad \text{and} \quad I_k \leq -\frac{k}{\sigma(k)} ,$$

and on the other hand [A]

$$(0.7) \quad S_k \leq C(k) \quad \text{and} \quad I_k \geq -C(k)$$

where, if $k = p_1 p_2 \dots p_r$ ($p_1 < p_2 < \dots < p_r$),

$$C(k) := \frac{p_1}{p_1 + 1} 2^{r-1} - \sum_{j=2}^r \frac{p_1 p_2 \dots p_{j-1}}{(p_1 + 1)(p_2 + 1) \dots (p_j + 1)} 2^{r-j} .$$

The purpose of this note is to improve on the estimates (0.6) for all k with $\omega(k) \geq 2$. As a preliminary we simplify the study of $E_k(x)$; in Section 1 we prove

THEOREM 1. *The function*

$$(0.8) \quad h(x) = h_k(x) := \frac{E_k(x)}{x}$$

¹⁾ The notation in (0.3) means that there are two unbounded positive sequences $\{x_i^+\}$ and $\{x_i^-\}$ ($i = 1, 2, \dots$), and two strictly positive constants C^+ and C^- , such that the inequalities $E_k(x_i^+) \geq C^+ x_i^+$ and $E_k(x_i^-) \leq -C^- x_i^-$ hold for each $i = 1, 2, \dots$.

possesses an asymptotic distribution function which is symmetric (and of bounded support). Moreover we have

$$(0.9) \quad I_k = -S_k.$$

Then we obtain in Section 2 a lower bound for S_k in the case where $k = pq$ ($p < q$ primes) which implies in particular

THEOREM 2. *For $k = 2q \geq 6$ where q is a prime we have*

$$(0.10) \quad S_k \geq \frac{q - \frac{1}{3}}{q + 1} = \frac{k}{\sigma(k)} + \frac{q - 1}{3(q + 1)}.$$

And finally in Section 3 we show

THEOREM 3. *Let $k = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$ are primes and $r \geq 2$, and let N be the positive integer such that*

$$(0.11) \quad \begin{aligned} f_r(p_2, \dots, p_r) &:= \left(\frac{\sigma(k/p_1)}{k/p_1} - 1 \right)^{-1} \\ &\in \begin{cases} (0, p_1^2 - 1) & (N = 1) \\ [p_1^N - 1, p_1^{N+1} - 1) & (N = 2, 3, \dots) \end{cases}. \end{aligned}$$

Then, except possibly in the case where $r = 2$, $p_1 = 2$ and $p_2 = 2^N - 1$, we have

$$(0.12) \quad \begin{aligned} S_k &\geq -(p_1^N - 1) \frac{k}{\sigma(k)} + \frac{(p_1^{2N} - 1)}{p_1^{N-1}(p_1 + 1)} \\ &\geq \frac{k}{\sigma(k)} + \frac{1}{(p_1 + 1)} \left(1 - \frac{1}{p_1^{N-1}} + \frac{1}{(\sigma(k/p_1) - k/p_1)p_1^{N+1} - 1} \right). \end{aligned}$$

We shall need the expression

$$(0.13) \quad h_k(x) = \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right) + o(1),$$

where the multiplicative arithmetical function γ_k is defined by

$$\gamma_k(p^m) = \begin{cases} 1 - p & \text{if } p \mid k, \\ 0 & \text{otherwise} \end{cases}$$

(see [HM, Theorem 1 and Lemma 1]), the fact that [HM, (4.1)], if we set

$$H(k, x) := \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

$$(0.14) \quad H(k, x) = H(k, [x]) - \frac{k}{\sigma(k)} \{x\} + o(1),$$

and

LEMMA 0. *We have*

$$(0.15) \quad S_k = \sup_{n \in \mathbf{Z}} H(k, n).$$

Proof. In view of (0.13), (0.14), and the definition of $H(k, x)$, it is sufficient to show that

$$(0.16) \quad \limsup_{N \rightarrow \infty, N \in \mathbf{N}} H(k, N) = \sup_{n \in \mathbf{Z}} H(k, n).$$

When $k = 1$ this is easily verified; when $k \geq 2$ and $N \in \mathbf{Z}$ we define for each positive integer i the positive integer $N_i := (|N| + 1)k^i + N$ and we see, since

$$(0.17) \quad \sum_{m \mid k^i} \frac{\gamma_k(m)}{m} \rightarrow 0 \quad (i \rightarrow \infty),$$

and since for every divisor m of k^i we have $\{N_i/m\} = \{N/m\}$, that

$$(0.18) \quad \lim_{i \rightarrow \infty} H(k, N_i) = H(k, N). \quad \square$$

1. PROOF OF THEOREM 1

We first set some terminology. Let $g: [1, \infty] \rightarrow \mathbf{R}$ be a measurable function, and consider as in [P1]

$$(1.1) \quad D_0(u) = D_{0,g}(u) := \lim_{x \rightarrow \infty} \frac{1}{x} \mu \{t \in [0, x], g(t) \leq u\},$$

and

$$(1.2) \quad D_0(u^+) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v), \quad D_0(u^-) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v),$$