Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 37 (1991)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k

Autor: PÉTERMANN, Y.-F. S.

Kapitel: 0. Introduction and statement of the results

DOI: https://doi.org/10.5169/seals-58739

L'Enseignement Mathématique, t. 37 (1991), p. 213-222

ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k

by Y.-F. S. Pétermann

Résumé. On étudie le comportement de la fonction bornée $h_k(x) := x^{-1}E_k(x)$, où $E_k(x) := \sum_{n \leqslant x} \delta_k(n) - (k/2\sigma(k))x^2$ est le terme irrégulier du comportement asymptotique moyen de $\delta_k(n)$, le plus grand diviseur de n premier à k (et où l'on peut sans perte supposer que k est sans facteur carré). On s'intéresse plus particulièrement aux nombres I_k et S_k, les lim inf et lim sup de $h_k(x)$ (lorsque $x \to \infty$), dont les valeurs exactes ne sont connues que si $k = 1$ ou si k est un nombre premier (Joshi et Vaidya [JV]). En établissant l'existence et la symétrie de la fonction de répartition de $h_k(n)$ (au sens de Wintner), on simplifie le problème en démontrant que $I_k = -S_k$. Puis, pour tous les k non premiers et sans facteur carré, on améliore explicitement l'estimation $S_k \geqslant k/\sigma(k)$ (de Herzog et Maxsein [HM], et indépendamment Adhikari, Balasubramanian et Sankaranarayanan [ABS]).

0. Introduction and statement of the results

For a fixed natural number k we denote by $\delta_k(n)$ the largest divisor of n which is prime to k. If κ is the squarefree core of k we have $\delta_k(n) = \delta_\kappa(n)$, and we shall assume from now on that k is squarefree. We define the associated error term

$$E_k(x) := \sum_{n \leqslant x} \delta_k(n) - \frac{k}{2\sigma(k)} x^2 , \tag{0.1}$$

where σ is the sum-of-divisors function. The behaviour of $E_k(x)$ has been investigated in [Su], [JV], [HM], [ABS], [AB], and very recently in [A]. It is known that [JV]

$$E_k(x) = O(x) \tag{0.2}$$

and that [JV, HM, ABS] [1])

$$E_k(x) = \Omega_\pm(x) . \tag{0.3}$$

However, the exact values of the lim sup and lim inf of $E_k(x)/x$ are not known, except in the special case where k is a prime p (and of course when $k = 1$); we have [JV]

$$\limsup_{x \to \infty} \frac{E_p(x)}{x} = \frac{p}{p+1} \quad \text{and} \quad \liminf_{x \to \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1} . \tag{0.4}$$

Let us from now on use the notation

$$S_k := \limsup_{x \to \infty} \frac{E_k(x)}{x} \quad \text{and} \quad I_k := \liminf_{x \to \infty} \frac{E_k(x)}{x} . \tag{0.5}$$

When the number $\omega(k)$ of (distinct) prime divisors of k exceeds 1, the best estimates known so far are on the one hand [HM, ABS]

$$S_k \geqslant \frac{k}{\sigma(k)} \quad \text{and} \quad I_k \leqslant -\frac{k}{\sigma(k)} , \tag{0.6}$$

and on the other hand [A]

$$S_k \leqslant C(k) \quad \text{and} \quad I_k \geqslant -C(k) \tag{0.7}$$

where, if $k = p_1 p_2 \dots p_r$ $(p_1 < p_2 < \dots < p_r)$,

$$C(k) := \frac{p_1}{p_1 + 1} 2^{r-1} - \sum_{j=2}^{r} \frac{p_1 p_2 \dots p_{j-1}}{(p_1+1)(p_2+1)\dots(p_j+1)} 2^{r-j} .$$

The purpose of this note is to improve on the estimates (0.6) for all k with $\omega(k) \geqslant 2$. As a preliminary we simplify the study of $E_k(x)$; in Section 1 we prove

THEOREM 1. *The function*

$$h(x) = h_k(x) := \frac{E_k(x)}{x} \tag{0.8}$$

[1]) The notation in (0.3) means that there are two unbounded positive sequences $\{x_i^+\}$ and $\{x_i^-\}$ $(i = 1, 2, \dots)$, and two strictly positive constants C^+ and C^-, such that the inequalities $E_k(x_i^+) \geqslant C^+ x_i^+$ and $E_k(x_i^-) \leqslant -C^- x_i^-$ hold for each $i = 1, 2, \dots$.

possesses an asymptotic distribution function which is symmetric (and of bounded support). Moreover we have

$$I_k = - S_k \,. \tag{0.9}$$

Then we obtain in Section 2 a lower bound for S_k in the case where $k = pq$ ($p < q$ primes) which implies in particular

THEOREM 2. *For* $k = 2q \geqslant 6$ *where* q *is a prime we have*

$$S_k \geqslant \frac{q - \frac{1}{3}}{q+1} = \frac{k}{\sigma(k)} + \frac{q-1}{3(q+1)} \,. \tag{0.10}$$

And finally in Section 3 we show

THEOREM 3. *Let* $k = p_1 p_2 \dots p_r$, *where* $p_1 < p_2 < \dots < p_r$ *are primes and* $r \geqslant 2$, *and let* N *be the positive integer such that*

$$\begin{aligned} f_r(p_2, \dots, p_r) : &= \left(\frac{\sigma(k/p_1}{k/p_1} - 1 \right)^{-1} \\ &\in \begin{cases} (0, p_1^2 - 1) & (N = 1) \\ [p_1^N - 1, p_1^{N+1} - 1) & (N = 2, 3, \dots) \,. \end{cases} \end{aligned} \tag{0.11}$$

Then, except possibly in the case where $r = 2, p_1 = 2$ *and* $p_2 = 2^N - 1$, *we have*

$$\begin{aligned} S_k &\geqslant - (p_1^N - 1) \frac{k}{\sigma(k)} + \frac{(p_1^{2N} - 1)}{p_1^{N-1}(p_1 + 1)} \\ &\geqslant \frac{k}{\sigma(k)} + \frac{1}{(p_1 + 1)} \left(1 - \frac{1}{p_1^{N-1}} + \frac{1}{(\sigma(k/p_1) - k/p_1) p_1^{N+1} - 1} \right) . \end{aligned} \tag{0.12}$$

We shall need the expression

$$h_k(x) = \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right) + o(1) \,, \tag{0.13}$$

where the multiplicative arithmetical function γ_k is defined by

$$\gamma_k(p^m) = \begin{cases} 1 - p & \text{if } p \mid k , \\ 0 & \text{otherwise} \end{cases}$$

(see [HM, Theorem 1 and Lemma 1]), the fact that [HM, (4.1)], if we set

$$H(k,x) := \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

$$H(k,x) = H(k,[x]) - \frac{k}{\sigma(k)} \{x\} + o(1) , \tag{0.14}$$

and

LEMMA 0. *We have*

$$S_k = \sup_{n \in \mathbf{Z}} H(k,n). \tag{0.15}$$

Proof. In view of (0.13), (0.14), and the definition of $H(k,x)$, it is sufficient to show that

$$\limsup_{N \to \infty, N \in \mathbf{N}} H(k,N) = \sup_{n \in \mathbf{Z}} H(k,n) . \tag{0.16}$$

When $k = 1$ this is easily verified; when $k \geqslant 2$ and $N \in \mathbf{Z}$ we define for each positive integer i the positive integer $N_i := (|N| + 1)k^i + N$ and we see, since

$$\sum_{m \nmid k^i} \frac{\gamma_k(m)}{m} \to 0 \ (i \to \infty) , \tag{0.17}$$

and since for every divisor m of k^i we have $\{N_i/m\} = \{N/m\}$, that

$$\lim_{i \to \infty} H(k,N_i) = H(k,N) . \quad \square \tag{0.18}$$

1. PROOF OF THEOREM 1

We first set some terminology. Let $g\colon [1, \infty] \to \mathbf{R}$ be a measurable function, and consider as in [P1]

$$D_0(u) = D_{0,g}(u) := \lim_{x \to \infty} \frac{1}{x} \mu \{ t \in [0,x], g(t) \leqslant x \} , \tag{1.1}$$

and

$$D_0(u^+) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) , \quad D_0(u^-) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) , \tag{1.2}$$