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**Autor:** PÉTERMANN, Y.-F. S.  
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ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR  
OF  $n$  PRIME TO A FIXED INTEGER  $k$

by Y.-F.S. PÉTERMANN

RÉSUMÉ. On étudie le comportement de la fonction bornée  $h_k(x) := x^{-1}E_k(x)$ , où  $E_k(x) := \sum_{n \leq x} \delta_k(n) - (k/2\sigma(k))x^2$  est le terme irrégulier du comportement asymptotique moyen de  $\delta_k(n)$ , le plus grand diviseur de  $n$  premier à  $k$  (et où l'on peut sans perte supposer que  $k$  est sans facteur carré). On s'intéresse plus particulièrement aux nombres  $I_k$  et  $S_k$ , les  $\liminf$  et  $\limsup$  de  $h_k(x)$  (lorsque  $x \rightarrow \infty$ ), dont les valeurs exactes ne sont connues que si  $k = 1$  ou si  $k$  est un nombre premier (Joshi et Vaidya [JV]). En établissant l'existence et la symétrie de la fonction de répartition de  $h_k(n)$  (au sens de Wintner), on simplifie le problème en démontrant que  $I_k = -S_k$ . Puis, pour tous les  $k$  non premiers et sans facteur carré, on améliore explicitement l'estimation  $S_k \geq k/\sigma(k)$  (de Herzog et Maxsein [HM], et indépendamment Adhikari, Balasubramanian et Sankaranarayanan [ABS]).

0. INTRODUCTION AND STATEMENT OF THE RESULTS

For a fixed natural number  $k$  we denote by  $\delta_k(n)$  the largest divisor of  $n$  which is prime to  $k$ . If  $\kappa$  is the squarefree core of  $k$  we have  $\delta_k(n) = \delta_\kappa(n)$ , and we shall assume from now on that  $k$  is squarefree. We define the associated error term

$$(0.1) \quad E_k(x) := \sum_{n \leq x} \delta_k(n) - \frac{k}{2\sigma(k)} x^2,$$

where  $\sigma$  is the sum-of-divisors function. The behaviour of  $E_k(x)$  has been investigated in [Su], [JV], [HM], [ABS], [AB], and very recently in [A]. It is known that [JV]

$$(0.2) \quad E_k(x) = O(x)$$

and that [JV, HM, ABS]<sup>1)</sup>

$$(0.3) \quad E_k(x) = \Omega_{\pm}(x).$$

However, the exact values of the lim sup and lim inf of  $E_k(x)/x$  are not known, except in the special case where  $k$  is a prime  $p$  (and of course when  $k = 1$ ); we have [JV]

$$(0.4) \quad \limsup_{x \rightarrow \infty} \frac{E_p(x)}{x} = \frac{p}{p+1} \quad \text{and} \quad \liminf_{x \rightarrow \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}.$$

Let us from now on use the notation

$$(0.5) \quad S_k := \limsup_{x \rightarrow \infty} \frac{E_k(x)}{x} \quad \text{and} \quad I_k := \liminf_{x \rightarrow \infty} \frac{E_k(x)}{x}.$$

When the number  $\omega(k)$  of (distinct) prime divisors of  $k$  exceeds 1, the best estimates known so far are on the one hand [HM, ABS]

$$(0.6) \quad S_k \geq \frac{k}{\sigma(k)} \quad \text{and} \quad I_k \leq -\frac{k}{\sigma(k)},$$

and on the other hand [A]

$$(0.7) \quad S_k \leq C(k) \quad \text{and} \quad I_k \geq -C(k)$$

where, if  $k = p_1 p_2 \dots p_r$  ( $p_1 < p_2 < \dots < p_r$ ),

$$C(k) := \frac{p_1}{p_1 + 1} 2^{r-1} - \sum_{j=2}^r \frac{p_1 p_2 \dots p_{j-1}}{(p_1 + 1)(p_2 + 1) \dots (p_j + 1)} 2^{r-j}.$$

The purpose of this note is to improve on the estimates (0.6) for all  $k$  with  $\omega(k) \geq 2$ . As a preliminary we simplify the study of  $E_k(x)$ ; in Section 1 we prove

**THEOREM 1.** *The function*

$$(0.8) \quad h(x) = h_k(x) := \frac{E_k(x)}{x}$$

<sup>1)</sup> The notation in (0.3) means that there are two unbounded positive sequences  $\{x_i^+\}$  and  $\{x_i^-\}$  ( $i = 1, 2, \dots$ ), and two strictly positive constants  $C^+$  and  $C^-$ , such that the inequalities  $E_k(x_i^+) \geq C^+ x_i^+$  and  $E_k(x_i^-) \leq -C^- x_i^-$  hold for each  $i = 1, 2, \dots$ .

possesses an asymptotic distribution function which is symmetric (and of bounded support). Moreover we have

$$(0.9) \quad I_k = -S_k.$$

Then we obtain in Section 2 a lower bound for  $S_k$  in the case where  $k = pq$  ( $p < q$  primes) which implies in particular

THEOREM 2. For  $k = 2q \geq 6$  where  $q$  is a prime we have

$$(0.10) \quad S_k \geq \frac{q - \frac{1}{3}}{q + 1} = \frac{k}{\sigma(k)} + \frac{q - 1}{3(q + 1)}.$$

And finally in Section 3 we show

THEOREM 3. Let  $k = p_1 p_2 \dots p_r$ , where  $p_1 < p_2 < \dots < p_r$  are primes and  $r \geq 2$ , and let  $N$  be the positive integer such that

$$(0.11) \quad \begin{aligned} f_r(p_2, \dots, p_r) &:= \left( \frac{\sigma(k/p_1)}{k/p_1} - 1 \right)^{-1} \\ &\in \begin{cases} (0, p_1^2 - 1) & (N = 1) \\ [p_1^N - 1, p_1^{N+1} - 1) & (N = 2, 3, \dots) \end{cases}. \end{aligned}$$

Then, except possibly in the case where  $r = 2$ ,  $p_1 = 2$  and  $p_2 = 2^N - 1$ , we have

$$(0.12) \quad \begin{aligned} S_k &\geq - (p_1^N - 1) \frac{k}{\sigma(k)} + \frac{(p_1^{2N} - 1)}{p_1^{N-1}(p_1 + 1)} \\ &\geq \frac{k}{\sigma(k)} + \frac{1}{(p_1 + 1)} \left( 1 - \frac{1}{p_1^{N-1}} + \frac{1}{(\sigma(k/p_1) - k/p_1)p_1^{N+1} - 1} \right). \end{aligned}$$

We shall need the expression

$$(0.13) \quad h_k(x) = \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left( \frac{1}{2} - \left\{ \frac{x}{n} \right\} \right) + o(1),$$

where the multiplicative arithmetical function  $\gamma_k$  is defined by

$$\gamma_k(p^m) = \begin{cases} 1 - p & \text{if } p \mid k, \\ 0 & \text{otherwise} \end{cases}$$

(see [HM, Theorem 1 and Lemma 1]), the fact that [HM, (4.1)], if we set

$$H(k, x) := \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left( \frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

$$(0.14) \quad H(k, x) = H(k, [x]) - \frac{k}{\sigma(k)} \{x\} + o(1),$$

and

LEMMA 0. *We have*

$$(0.15) \quad S_k = \sup_{n \in \mathbf{Z}} H(k, n).$$

*Proof.* In view of (0.13), (0.14), and the definition of  $H(k, x)$ , it is sufficient to show that

$$(0.16) \quad \limsup_{N \rightarrow \infty, N \in \mathbf{N}} H(k, N) = \sup_{n \in \mathbf{Z}} H(k, n).$$

When  $k = 1$  this is easily verified; when  $k \geq 2$  and  $N \in \mathbf{Z}$  we define for each positive integer  $i$  the positive integer  $N_i := (|N| + 1)k^i + N$  and we see, since

$$(0.17) \quad \sum_{m \nmid k^i} \frac{\gamma_k(m)}{m} \rightarrow 0 \quad (i \rightarrow \infty),$$

and since for every divisor  $m$  of  $k^i$  we have  $\{N_i/m\} = \{N/m\}$ , that

$$(0.18) \quad \lim_{i \rightarrow \infty} H(k, N_i) = H(k, N). \quad \square$$

## 1. PROOF OF THEOREM 1

We first set some terminology. Let  $g: [1, \infty] \rightarrow \mathbf{R}$  be a measurable function, and consider as in [P1]

$$(1.1) \quad D_0(u) = D_{0,g}(u) := \lim_{x \rightarrow \infty} \frac{1}{x} \mu \{t \in [0, x], g(t) \leq x\},$$

and

$$(1.2) \quad D_0(u^+) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v), \quad D_0(u^-) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v),$$