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COROLLARY 2.9. *As representations of* Y, *we have*

$$W_m(a)^* \cong W_m(-a) .$$

Proof. On $W_m(a)$, $J(x)$ acts as ax. Therefore, on $W_m(a)^*$, $J(x)$ acts as $-ax$.

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation* $W_m(a)$ *has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form $< , >$ on $W_m(a)$ such that

$$<y.v_1, v_2> = <v_1, \omega(y).v_2> \tag{2.11}$$

for all $y \in Y$, $v_1, v_2 \in W_m(a)$.

Proof. It is well-known that the representation W_m of $\mathfrak{sl}_2$ carries a form $< , >$ which satisfies (2.11) for all $y \in \mathfrak{sl}_2$. Moreover, the form is unique up to a scalar multiple because W_m is irreducible. To prove (2.11) in general, it suffices to check the case $y = x_k^+$, since the case $y = x_k^-$ then follows because $< , >$ is symmetric, and $\omega(x_k^+) = x_k^-$. Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$<x_k^+.e_i, e_{i+k}> = <e_i, x_k^-.e_{i+k}> \tag{2.12}$$

(with the understanding that $e_i = 0$ unless $0 \leqslant i \leqslant n$). This follows easily from Proposition 2.6 and the invariance of $< , >$ under $\mathfrak{sl}_2$.

3. A COMBINATORIAL INTERLUDE

The form of the polynomial P associated to the representation $W_m(a)$ in Corollary 2.7(b) suggests the following definition.

Definition 3.1. A non-empty finite set of complex numbers is said to be a *string* if it is of the form $\{a, a+1, \ldots, a+n\}$ for some $a \in \mathbf{C}$ and some $n \in \mathbf{N}$. The centre of the string is $a + \frac{n}{2}$ and its length is $n+1$.

We shall also need:

Definition 3.2. Two strings S_1 and S_2 are said to be *non-interacting* if either

(1) $S_1 \cup S_2$ is not a string, or

(2) $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

Remark. We shall discuss the "interactions" of strings in section 4.

We should like to assert that the set of roots of an arbitrary polynomial is a union of non-interacting strings. To make this precise, we need one last definition.

Definition 3.3. A *set with multiplicities* is a map $f: \Sigma \to \mathbf{N}$, where Σ is a set. If Σ is a finite set, the *cardinality* of f is

$$| f | = \sum_{x \in \Sigma} f(x) .$$

The *union* of two sets with multiplicities is the sum of the corresponding maps. Note that any set is a set with multiplicities, all values of the map being equal to one. Also, the roots of a polynomial $P \in \mathbf{C}[u]$ form a set with multiplicities in a natural way. In particular, the roots of the polynomial associated to $W_m(a)$ in Corollary 2.7 (b) form a single string

$$S_m(a) = \left\{ a - \frac{1}{2}m + \frac{1}{2}, \ldots, a + \frac{1}{2}m - \frac{1}{2} \right\}$$

with centre a and length m.

We shall need the following simple result whose verification we leave to the reader.

LEMMA 3.4. *Two strings $S_m(a)$ and $S_n(b)$ are non-interacting if and only if it is not true that*

$$| a - b | = \frac{1}{2}(m + n) , \frac{1}{2}(m + n) - 1, \ldots, \quad or \quad \frac{1}{2} | m - n | + 1 .$$

The result we want is:

PROPOSITION 3.5. *Any finite set of complex numbers with multiplicities can be written uniquely as a union of strings, any two of which are non-interacting.*

Proof. Let $f: \Sigma \to \mathbf{N}$ be a finite set of complex numbers with multiplicities. The proof is by induction on $| f |$. If $| f | = 0$ or 1 there is nothing to prove.

Choose $s \in \Sigma$, let S be the maximal string of numbers in Σ which contains s, and let g be the characteristic function of S. By induction, $f - g$ is a union of non-interacting strings. If T is any such string, then S and T are non-interacting, since if $T \nsubseteq S$ then $S \cup T$ cannot be a string, by maximality of S. Thus, adjoining S to the string decomposition of $f - g$ gives the desired decomposition of f.

As for uniqueness, we first show that the string S above must occur in any decomposition of f as a union of non-interacting strings. For, otherwise, let T be a maximal string in such a decomposition which contains s. Then T is properly contained in S, so there exists $u \in S - T$ such that $T \cup \{u\}$ is a string. Let U be a string in the given decomposition of f which contains u. Then, by its maximality, T cannot be contained in U, so T and U are interacting, a contradiction.

Thus, S must occur in any two decompositions of f as a union of non-interacting strings. Deleting S from both decompositions and using the induction hypothesis, one deduces that the two decompositions are the same.

We conclude this section with the computation of a determinant which plays the same role for Yangians as the Vandermonde determinant plays in the classification of integrable representations of affine Lie algebras [1].

Let r be a positive integer and let b_j, m_j, $1 \leqslant j \leqslant r$, be complex numbers. Quantities $d_{k,j}, A_{k,j}$ for $1 \leqslant j \leqslant r$, $0 \leqslant k \leqslant r - 1$, are defined inductively by the following formulas:

$$\begin{aligned} A_{k,j} &= b_j^k + b_j^{k-1} d_{0,j} + \cdots + d_{k-1,j} \\ d_{k,j} &= m_{j+1} A_{k,j+1} + d_{k,j+1}, \qquad d_{k,r} = 0 \end{aligned} \tag{3.6}$$

(we set $d_{k,r+1} = 0$). Let A be the matrix $(A_{k,j})$ with $1 \leqslant j \leqslant r$, $0 \leqslant k \leqslant r - 1$.

PROPOSITION 3.7. $\det A = \prod_{1 \leqslant k < j \leqslant r} (b_j - b_k - m_j)$.

Remark. One can think of $\det A$ as a "quantum Vandermonde determinant". Indeed, recall that Y is obtained from a deformation of $U(\mathfrak{sl}_2[t])$ by setting the deformation parameter h equal to one. If we had not set $h = 1$, then in equation (3.6) $d_{k,j}$ would be replaced by $hd_{k,j}$ and in equation (3.7) m_j would be replaced by hm_j. Thus, in the "classical limit" $h \to 0$, $\det A$ becomes the usual Vandermonde determinant and (3.7) its well-known factorization.

Our proof of (3.7) is rather indirect and will be given in the next section.