

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** YANGIANS AND R-MATRICES  
**Autor:** Chari, Vyjayanthi / Pressley, Andrew  
**Kapitel:** 2. Highest weight representations  
**DOI:** <https://doi.org/10.5169/seals-57909>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 07.03.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

all  $k, l, m \in \mathbf{Z}_+$ . If  $u = u_1 u_2 \dots u_n$  is any monomial in the generators of degree  $n$ , define its index

$$\text{ind}(u) = \sum_{i < j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if } u_i < u_j \\ 1 & \text{if } u_j < u_i. \end{cases}$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

## 2. HIGHEST WEIGHT REPRESENTATIONS

By analogy with the definition of highest weight representations of semi-simple Lie algebras, one makes the following

*Definition 2.1.* A representation  $V$  of the Yangian  $Y$  is said to be *highest weight* if there is a vector  $\Omega \in V$  such that  $V = Y\Omega$  and

$$x_k^+ \Omega = 0, \quad h_k \Omega = d_k \Omega, \quad k = 0, 1, \dots$$

for some sequence of complex numbers  $\mathbf{d} = (d_0, d_1, \dots)$ . In this case,  $\Omega$  is called a highest weight vector of  $V$  and  $\mathbf{d}$  its highest weight.

*Remark.* It follows immediately from Definition 1.1 that the assignment  $x \mapsto x$  for  $x \in \mathfrak{sl}_2$  extends to a homomorphism of algebras  $\iota: U(\mathfrak{sl}_2) \rightarrow Y$ . By Proposition 2.5 below,  $\iota$  is injective. Thus, any representation of  $Y$  can be restricted to give a representation of  $\mathfrak{sl}_2$ . In particular, we can speak of weights relative to  $\mathfrak{sl}_2$  as well as relative to  $Y$ . It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of  $Y$  of any given highest weight:

*Definition 2.2.* Let  $\mathbf{d} = (d_0, d_1, \dots)$  be any sequence of complex numbers. The *Verma representation*  $M(\mathbf{d})$  is the quotient of  $Y$  by the left ideal generated by  $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbf{Z}_+}$ .

**PROPOSITION 2.3.** *The Verma representation  $M(\mathbf{d})$  is a highest weight representation with highest weight  $\mathbf{d}$ , and every such representation is*

isomorphic to a quotient of  $M(\mathbf{d})$ . Moreover,  $M(\mathbf{d})$  has a unique irreducible quotient  $V(\mathbf{d})$ .

*Proof.* Only the last statement requires proof. We consider  $M(\mathbf{d})$  as a representation of  $\mathfrak{sl}_2$ . By Proposition 1.11, the  $d_0$ -weight space  $\{v \in M(\mathbf{d}) : h_0.v = d_0v\}$  is one-dimensional, and spanned by the highest weight vector  $1 \in M(\mathbf{d})$ . Thus, if  $M_1$  and  $M_2$  are two proper subrepresentations of  $M(\mathbf{d})$ , then  $M_1 + M_2$  is also proper. It follows that  $M(\mathbf{d})$  has a unique maximal proper subrepresentation.

The question of which highest weight representations are finite-dimensional was answered by Drinfel'd in [5, Theorem 2]. His result may be stated as follows.

**THEOREM 2.4.** (a) *Every irreducible finite-dimensional representation of  $Y$  is highest weight.*

(b) *The irreducible highest weight representation  $V(\mathbf{d})$  of  $Y$  is finite-dimensional if and only if there exists a monic polynomial  $P \in \mathbf{C}[u]$  such that*

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

*in the sense that the right-hand side is the Laurent expansion of the left-hand side about  $u = \infty$ .*

To construct examples of highest weight representations of  $Y$ , we need the following result, which is an immediate consequence of the defining relations (1.1).

**PROPOSITION 2.5.** (a) *The assignment  $x \mapsto x, J(x) \mapsto 0$  extends to a homomorphism of algebras  $\varepsilon_0 : Y \rightarrow U(\mathfrak{sl}_2)$ .*

(b) *For any  $a \in \mathbf{C}$ , the assignment  $x \mapsto x, J(x) \mapsto J(x) + ax$  extends to an automorphism  $\tau_a$  of  $Y$ .*

By part (a), if  $V$  is a representation of  $\mathfrak{sl}_2$ , one can pull it back by  $\varepsilon_0$  to give a representation  $V$  of  $Y$ . Pulling back this representation by  $\tau_a$  then gives a one-parameter family of representations  $V(a)$  of  $Y$ . Note that  $V(a)$  is an irreducible representation of  $Y$  because  $\varepsilon_0$  is surjective.

Let  $W_m$  be the  $(m+1)$ -dimensional irreducible representation of  $\mathfrak{sl}_2, m \in \mathbf{Z}_+$ . Then,  $W_m(a)$  has a basis  $\{e_0, \dots, e_m\}$  on which the action of  $Y$  is given by:

$$x^+ . e_i = (i+1)e_{i+1}, \quad x^- . e_i = (m-i+1)e_{i-1}, \quad h . e_i = (2i-m)e_i,$$

the action of  $J(h)$  (resp.  $J(x^\pm)$ ) being  $a$  times that of  $h$  (resp.  $x^\pm$ ). To make contact with the theory of highest weight representations, we need:

PROPOSITION 2.6. *The action of the generators  $h_k, x_k^\pm$  on  $W_m(a)$  is given by:*

$$\begin{aligned} (1) \quad x_k^+ \cdot e_i &= \left( a - \frac{1}{2}m + i + \frac{1}{2} \right)^k (i+1)e_{i+1}; \\ (2) \quad x_k^- \cdot e_i &= \left( a - \frac{1}{2}m + i - \frac{1}{2} \right)^k (m-i+1)e_{i-1}; \\ (3) \quad h_k \cdot e_i &= \left\{ \left( a - \frac{1}{2}m + i - \frac{1}{2} \right)^k i(m-i+1) \right. \\ &\quad \left. - \left( a - \frac{1}{2}m + i + \frac{1}{2} \right)^k (i+1)(m-i) \right\} e_i. \end{aligned}$$

*Proof.* It is straightforward to check, using the relations (1)-(3) in Theorem 1.2, that these formulas do define a representation of  $Y$ . It therefore suffices to check that they also give the correct action of the generators  $h, J(h), x^\pm, J(x^\pm)$ . This is another straightforward computation, using the isomorphism  $\phi$  in (1.2).

COROLLARY 2.7. (a)  $W_m(a)$  is a highest weight representation with highest weight  $\mathbf{d} = (d_0, d_1, \dots)$  given by

$$d_k = m \left( a + \frac{1}{2}m - \frac{1}{2} \right)^k.$$

(b) The monic polynomial  $P$  associated to  $W_m(a)$  is given by

$$P(u) = \left( u - a + \frac{1}{2}m - \frac{1}{2} \right) \left( u - a + \frac{1}{2}m - \frac{3}{2} \right) \dots \left( u - a - \frac{1}{2}m + \frac{1}{2} \right).$$

*Proof.* (a) It is clear that  $e_m$  is a highest weight vector for  $W_m(a)$  relative to  $Y$ . The eigenvalues of the  $h_k$  on  $e_m$  are as stated.

(b) By Theorem 2.4(b), the polynomial  $P$  is determined by

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} m \left( a + \frac{1}{2}m - \frac{1}{2} \right)^k u^{-k-1}$$

$$= \frac{\left( u - a + \frac{1}{2}m + \frac{1}{2} \right)}{\left( u - a - \frac{1}{2}m + \frac{1}{2} \right)} .$$

The stated  $P$  clearly satisfies this equation.

In section 4 we shall need to consider the duals of the evaluation representations  $W_m(a)$ . If  $V$  is any finite-dimensional representation of  $Y$ , its dual  $V^*$  is naturally a representation of  $Y^{op}$ , the vector space  $Y$  with the opposite multiplication:

$$x \cdot y \text{ (in } Y^{op}\text{)} = y \cdot x \text{ (in } Y\text{)} .$$

Moreover,  $Y^{op}$  is a Hopf algebra with the same co-multiplication as  $Y$ .

PROPOSITION 2.8. *There is an isomorphism of Hopf algebras  $\theta: Y \rightarrow Y^{op}$  such that*

$$\theta(x) = -x, \quad \theta(J(x)) = J(x)$$

for all  $x \in \mathfrak{sl}_2$ .

*Proof.* It is sufficient to prove that the assignment  $x \mapsto -x, J(x) \mapsto J(x)$  extends to a homomorphism of Hopf algebras  $Y \rightarrow Y^{op}$ . The relations in  $Y^{op}$  are obtained by inserting a minus sign on the right-hand side of relations (1) and (3) in (1.1). The result is now clear.

*Remark.* The anti-homomorphism  $\theta: Y \rightarrow Y$  is closely related to the antipode  $S$  of  $Y$ , which is given by

$$S(x) = -x, \quad S(J(x)) = -J(x) + \frac{1}{4}cx,$$

where  $c$  is the eigenvalue of the Casimir operator in the adjoint representation of  $\mathfrak{sl}_2$  (which depends of course on the choice of inner product  $(,)$  on  $\mathfrak{sl}_2$ ).

Thus, if  $V$  is a finite-dimensional representation of  $Y$ , then  $V^*$  is a representation of  $Y$  with action

$$(y \cdot f)(v) = f(\theta(y) \cdot v),$$

for  $y \in Y, v \in V$  and  $f \in V^*$ . Moreover, the fact that  $\theta$  preserves the co-multiplication implies that  $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$  for any two representations  $V_1, V_2$  of  $Y$ .

COROLLARY 2.9. *As representations of  $Y$ , we have*

$$W_m(a)^* \cong W_m(-a).$$

*Proof.* On  $W_m(a)$ ,  $J(x)$  acts as  $ax$ . Therefore, on  $W_m(a)^*$ ,  $J(x)$  acts as  $-ax$ .

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation  $W_m(a)$  has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $W_m(a)$  such that

$$(2.11) \quad \langle y \cdot v_1, v_2 \rangle = \langle v_1, \omega(y) \cdot v_2 \rangle$$

for all  $y \in Y$ ,  $v_1, v_2 \in W_m(a)$ .

*Proof.* It is well-known that the representation  $W_m$  of  $\mathfrak{sl}_2$  carries a form  $\langle \cdot, \cdot \rangle$  which satisfies (2.11) for all  $y \in \mathfrak{sl}_2$ . Moreover, the form is unique up to a scalar multiple because  $W_m$  is irreducible. To prove (2.11) in general, it suffices to check the case  $y = x_k^+$ , since the case  $y = x_k^-$  then follows because  $\langle \cdot, \cdot \rangle$  is symmetric, and  $\omega(x_k^+) = x_k^-$ . Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$(2.12) \quad \langle x_k^+ \cdot e_i, e_{i+k} \rangle = \langle e_i, x_k^- \cdot e_{i+k} \rangle$$

(with the understanding that  $e_i = 0$  unless  $0 \leq i \leq n$ ). This follows easily from Proposition 2.6 and the invariance of  $\langle \cdot, \cdot \rangle$  under  $\mathfrak{sl}_2$ .

### 3. A COMBINATORIAL INTERLUDE

The form of the polynomial  $P$  associated to the representation  $W_m(a)$  in Corollary 2.7(b) suggests the following definition.

*Definition 3.1.* A non-empty finite set of complex numbers is said to be a *string* if it is of the form  $\{a, a+1, \dots, a+n\}$  for some  $a \in \mathbf{C}$  and some  $n \in \mathbf{N}$ .

The centre of the string is  $a + \frac{n}{2}$  and its length is  $n+1$ .

We shall also need:

*Definition 3.2.* Two strings  $S_1$  and  $S_2$  are said to be *non-interacting* if either