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all  $k, l, m \in \mathbf{Z}_+$ . If  $u = u_1 u_2 \dots u_n$  is any monomial in the generators of degree  $n$ , define its index

$$\text{ind}(u) = \sum_{i < j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if } u_i < u_j \\ 1 & \text{if } u_j < u_i. \end{cases}$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

## 2. HIGHEST WEIGHT REPRESENTATIONS

By analogy with the definition of highest weight representations of semi-simple Lie algebras, one makes the following

*Definition 2.1.* A representation  $V$  of the Yangian  $Y$  is said to be *highest weight* if there is a vector  $\Omega \in V$  such that  $V = Y\Omega$  and

$$x_k^+ \Omega = 0, \quad h_k \Omega = d_k \Omega, \quad k = 0, 1, \dots$$

for some sequence of complex numbers  $\mathbf{d} = (d_0, d_1, \dots)$ . In this case,  $\Omega$  is called a highest weight vector of  $V$  and  $\mathbf{d}$  its highest weight.

*Remark.* It follows immediately from Definition 1.1 that the assignment  $x \mapsto x$  for  $x \in \mathfrak{sl}_2$  extends to a homomorphism of algebras  $\iota: U(\mathfrak{sl}_2) \rightarrow Y$ . By Proposition 2.5 below,  $\iota$  is injective. Thus, any representation of  $Y$  can be restricted to give a representation of  $\mathfrak{sl}_2$ . In particular, we can speak of weights relative to  $\mathfrak{sl}_2$  as well as relative to  $Y$ . It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of  $Y$  of any given highest weight:

*Definition 2.2.* Let  $\mathbf{d} = (d_0, d_1, \dots)$  be any sequence of complex numbers. The *Verma representation*  $M(\mathbf{d})$  is the quotient of  $Y$  by the left ideal generated by  $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbf{Z}_+}$ .

**PROPOSITION 2.3.** *The Verma representation  $M(\mathbf{d})$  is a highest weight representation with highest weight  $\mathbf{d}$ , and every such representation is*

isomorphic to a quotient of  $M(\mathbf{d})$ . Moreover,  $M(\mathbf{d})$  has a unique irreducible quotient  $V(\mathbf{d})$ .

*Proof.* Only the last statement requires proof. We consider  $M(\mathbf{d})$  as a representation of  $\mathfrak{sl}_2$ . By Proposition 1.11, the  $d_0$ -weight space  $\{v \in M(\mathbf{d}) : h_0.v = d_0 v\}$  is one-dimensional, and spanned by the highest weight vector  $1 \in M(\mathbf{d})$ . Thus, if  $M_1$  and  $M_2$  are two proper subrepresentations of  $M(\mathbf{d})$ , then  $M_1 + M_2$  is also proper. It follows that  $M(\mathbf{d})$  has a unique maximal proper subrepresentation.

The question of which highest weight representations are finite-dimensional was answered by Drinfel'd in [5, Theorem 2]. His result may be stated as follows.

**THEOREM 2.4.** (a) *Every irreducible finite-dimensional representation of  $Y$  is highest weight.*

(b) *The irreducible highest weight representation  $V(\mathbf{d})$  of  $Y$  is finite-dimensional if and only if there exists a monic polynomial  $P \in \mathbb{C}[u]$  such that*

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

*in the sense that the right-hand side is the Laurent expansion of the left-hand side about  $u = \infty$ .*

To construct examples of highest weight representations of  $Y$ , we need the following result, which is an immediate consequence of the defining relations (1.1).

**PROPOSITION 2.5.** (a) *The assignment  $x \mapsto x, J(x) \mapsto 0$  extends to a homomorphism of algebras  $\varepsilon_0 : Y \rightarrow U(\mathfrak{sl}_2)$ .*

(b) *For any  $a \in \mathbb{C}$ , the assignment  $x \mapsto x, J(x) \mapsto J(x) + ax$  extends to an automorphism  $\tau_a$  of  $Y$ .*

By part (a), if  $V$  is a representation of  $\mathfrak{sl}_2$ , one can pull it back by  $\varepsilon_0$  to give a representation  $V$  of  $Y$ . Pulling back this representation by  $\tau_a$  then gives a one-parameter family of representations  $V(a)$  of  $Y$ . Note that  $V(a)$  is an irreducible representation of  $Y$  because  $\varepsilon_0$  is surjective.

Let  $W_m$  be the  $(m+1)$ -dimensional irreducible representation of  $\mathfrak{sl}_2$ ,  $m \in \mathbb{Z}_+$ . Then,  $W_m(a)$  has a basis  $\{e_0, \dots, e_m\}$  on which the action of  $Y$  is given by:

$$x^+ . e_i = (i+1)e_{i+1}, \quad x^- . e_i = (m-i+1)e_{i-1}, \quad h . e_i = (2i-m)e_i,$$

the action of  $J(h)$  (resp.  $J(x^\pm)$ ) being  $a$  times that of  $h$  (resp.  $x^\pm$ ). To make contact with the theory of highest weight representations, we need:

PROPOSITION 2.6. *The action of the generators  $h_k, x_k^\pm$  on  $W_m(a)$  is given by:*

$$\begin{aligned} (1) \quad x_k^+ \cdot e_i &= \left( a - \frac{1}{2}m + i + \frac{1}{2} \right)^k (i+1) e_{i+1}; \\ (2) \quad x_k^- \cdot e_i &= \left( a - \frac{1}{2}m + i - \frac{1}{2} \right)^k (m-i+1) e_{i-1}; \\ (3) \quad h_k \cdot e_i &= \left\{ \left( a - \frac{1}{2}m + i - \frac{1}{2} \right)^k i(m-i+1) \right. \\ &\quad \left. - \left( a - \frac{1}{2}m + i + \frac{1}{2} \right)^k (i+1)(m-i) \right\} e_i. \end{aligned}$$

*Proof.* It is straightforward to check, using the relations (1)-(3) in Theorem 1.2, that these formulas do define a representation of  $Y$ . It therefore suffices to check that they also give the correct action of the generators  $h, J(h), x^\pm, J(x^\pm)$ . This is another straightforward computation, using the isomorphism  $\phi$  in (1.2).

COROLLARY 2.7. (a)  $W_m(a)$  is a highest weight representation with highest weight  $\mathbf{d} = (d_0, d_1, \dots)$  given by

$$d_k = m \left( a + \frac{1}{2}m - \frac{1}{2} \right)^k.$$

(b) The monic polynomial  $P$  associated to  $W_m(a)$  is given by

$$P(u) = \left( u - a + \frac{1}{2}m - \frac{1}{2} \right) \left( u - a + \frac{1}{2}m - \frac{3}{2} \right) \dots \left( u - a - \frac{1}{2}m + \frac{1}{2} \right).$$

*Proof.* (a) It is clear that  $e_m$  is a highest weight vector for  $W_m(a)$  relative to  $Y$ . The eigenvalues of the  $h_k$  on  $e_m$  are as stated.

(b) By Theorem 2.4(b), the polynomial  $P$  is determined by

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} m \left( a + \frac{1}{2}m - \frac{1}{2} \right)^k u^{-k-1}$$

$$= \frac{\left(u - a + \frac{1}{2}m + \frac{1}{2}\right)}{\left(u - a - \frac{1}{2}m + \frac{1}{2}\right)} .$$

The stated  $P$  clearly satisfies this equation.

In section 4 we shall need to consider the duals of the evaluation representations  $W_m(a)$ . If  $V$  is any finite-dimensional representation of  $Y$ , its dual  $V^*$  is naturally a representation of  $Y^{op}$ , the vector space  $Y$  with the opposite multiplication:

$$x \cdot y \text{ (in } Y^{op}) = y \cdot x \text{ (in } Y) .$$

Moreover,  $Y^{op}$  is a Hopf algebra with the same co-multiplication as  $Y$ .

**PROPOSITION 2.8.** *There is an isomorphism of Hopf algebras  $\theta: Y \rightarrow Y^{op}$  such that*

$$\theta(x) = -x, \quad \theta(J(x)) = J(x)$$

for all  $x \in \mathfrak{sl}_2$ .

*Proof.* It is sufficient to prove that the assignment  $x \mapsto -x$ ,  $J(x) \mapsto J(x)$  extends to a homomorphism of Hopf algebras  $Y \rightarrow Y^{op}$ . The relations in  $Y^{op}$  are obtained by inserting a minus sign on the right-hand side of relations (1) and (3) in (1.1). The result is now clear.

*Remark.* The anti-homomorphism  $\theta: Y \rightarrow Y$  is closely related to the antipode  $S$  of  $Y$ , which is given by

$$S(x) = -x, \quad S(J(x)) = -J(x) + \frac{1}{4}cx,$$

where  $c$  is the eigenvalue of the Casimir operator in the adjoint representation of  $\mathfrak{sl}_2$  (which depends of course on the choice of inner product  $(\cdot, \cdot)$  on  $\mathfrak{sl}_2$ ).

Thus, if  $V$  is a finite-dimensional representation of  $Y$ , then  $V^*$  is a representation of  $Y$  with action

$$(y \cdot f)(v) = f(\theta(y) \cdot v),$$

for  $y \in Y$ ,  $v \in V$  and  $f \in V^*$ . Moreover, the fact that  $\theta$  preserves the co-multiplication implies that  $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$  for any two representations  $V_1, V_2$  of  $Y$ .

COROLLARY 2.9. *As representations of  $Y$ , we have*

$$W_m(a)^* \cong W_m(-a).$$

*Proof.* On  $W_m(a)$ ,  $J(x)$  acts as  $ax$ . Therefore, on  $W_m(a)^*$ ,  $J(x)$  acts as  $-ax$ .

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation  $W_m(a)$  has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $W_m(a)$  such that

$$(2.11) \quad \langle y \cdot v_1, v_2 \rangle = \langle v_1, \omega(y) \cdot v_2 \rangle$$

for all  $y \in Y$ ,  $v_1, v_2 \in W_m(a)$ .

*Proof.* It is well-known that the representation  $W_m$  of  $\mathfrak{sl}_2$  carries a form  $\langle \cdot, \cdot \rangle$  which satisfies (2.11) for all  $y \in \mathfrak{sl}_2$ . Moreover, the form is unique up to a scalar multiple because  $W_m$  is irreducible. To prove (2.11) in general, it suffices to check the case  $y = x_k^+$ , since the case  $y = x_k^-$  then follows because  $\langle \cdot, \cdot \rangle$  is symmetric, and  $\omega(x_k^+) = x_k^-$ . Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$(2.12) \quad \langle x_k^+ \cdot e_i, e_{i+k} \rangle = \langle e_i, x_k^- \cdot e_{i+k} \rangle$$

(with the understanding that  $e_i = 0$  unless  $0 \leq i \leq n$ ). This follows easily from Proposition 2.6 and the invariance of  $\langle \cdot, \cdot \rangle$  under  $\mathfrak{sl}_2$ .

### 3. A COMBINATORIAL INTERLUDE

The form of the polynomial  $P$  associated to the representation  $W_m(a)$  in Corollary 2.7(b) suggests the following definition.

*Definition 3.1.* A non-empty finite set of complex numbers is said to be a *string* if it is of the form  $\{a, a+1, \dots, a+n\}$  for some  $a \in \mathbb{C}$  and some  $n \in \mathbb{N}$ .

The centre of the string is  $a + \frac{n}{2}$  and its length is  $n+1$ .

We shall also need:

*Definition 3.2.* Two strings  $S_1$  and  $S_2$  are said to be *non-interacting* if either