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all k, l,  $m \in \mathbb{Z}_+$ . If  $u = u_1 u_2 \dots u_n$  is any monomial in the generators of degree n, define its index

$$\operatorname{ind}(u) = \sum_{i < j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if} & u_i < u_j \\ 1 & \text{if} & u_i < u_i \end{cases}.$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

## 2. Highest weight representations

By analogy with the definition of highest weight representations of semisimple Lie algebras, one makes the following

Definition 2.1. A representation V of the Yangian Y is said to be highest weight if there is a vector  $\Omega \in V$  such that  $V = Y\Omega$  and

$$x_k^+\Omega=0$$
,  $h_k\Omega=d_k\Omega$ ,  $k=0,1,...$ 

for some sequence of complex numbers  $\mathbf{d} = (d_0, d_1, ...)$ . In this case,  $\Omega$  is called a highest weight vector of V and  $\mathbf{d}$  its highest weight.

Remark. It follows immediately from Definition 1.1 that the assignment  $x \mapsto x$  for  $x \in \mathfrak{gl}_2$  extends to a homomorphism of algebras  $\mathfrak{l}: U(\mathfrak{gl}_2) \to Y$ . By Proposition 2.5 below,  $\mathfrak{l}$  is injective. Thus, any representation of Y can be restricted to give a representation of  $\mathfrak{gl}_2$ . In particular, we can speak of weights relative to  $\mathfrak{gl}_2$  as well as relative to Y. It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of Y of any given highest weight:

Definition 2.2. Let  $\mathbf{d} = (d_0, d_1, ...)$  be any sequence of complex numbers. The Verma representation  $M(\mathbf{d})$  is the quotient of Y by the left ideal generated by  $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbb{Z}_+}$ .

PROPOSITION 2.3. The Verma representation  $M(\mathbf{d})$  is a highest weight representation with highest weight  $\mathbf{d}$ , and every such representation is

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isomorphic to a quotient of  $M(\mathbf{d})$ . Moreover,  $M(\mathbf{d})$  has a unique irreducible quotient  $V(\mathbf{d})$ .

**Proof.** Only the last statement requires proof. We consider  $M(\mathbf{d})$  as a representation of  $\mathfrak{Sl}_2$ . By Proposition 1.11, the  $d_0$ -weight space  $\{v \in M(\mathbf{d}): h_0.v = d_0v\}$  is one-dimensional, and spanned by the highest weight vector  $1 \in M(\mathbf{d})$ . Thus, if  $M_1$  and  $M_2$  are two proper subrepresentations of  $M(\mathbf{d})$ , then  $M_1 + M_2$  is also proper. It follows that  $M(\mathbf{d})$  has a unique maximal proper subrepresentation.

The question of which highest weight representations are finite-dimensional was answered by Drinfel'd in [5, Theorem 2]. His result may be stated as follows.

Theorem 2.4. (a) Every irreducible finite-dimensional representation of Y is highest weight.

(b) The irreducible highest weight representation  $V(\mathbf{d})$  of Y is finite-dimensional if and only if there exists a monic polynomial  $P \in \mathbb{C}[u]$  such that

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about  $u = \infty$ .

To construct examples of highest weight representations of Y, we need the following result, which is an immediate consequence of the defining relations (1.1).

PROPOSITION 2.5. (a) The assignment  $x \mapsto x$ ,  $J(x) \mapsto 0$  extends to a homomorphism of algebras  $\varepsilon_0: Y \to U(\mathfrak{sl}_2)$ .

(b) For any  $a \in \mathbb{C}$ , the assignment  $x \mapsto x$ ,  $J(x) \mapsto J(x) + ax$  extends to an automorphism  $\tau_a$  of Y.

By part (a), if V is a representation of  $\mathfrak{Sl}_2$ , one can pull it back by  $\mathfrak{E}_0$  to give a representation V of Y. Pulling back this representation by  $\tau_a$  then gives a one-parameter family of representations V(a) of Y. Note that V(a) is an irreducible representation of Y because  $\mathfrak{E}_0$  is surjective.

Let  $W_m$  be the (m+1)-dimensional irreducible representation of  $\mathfrak{Sl}_2$ ,  $m \in \mathbb{Z}_+$ . Then,  $W_m(a)$  has a basis  $\{e_0, ..., e_m\}$  on which the action of Y is given by:

$$x^+ \cdot e_i = (i+1)e_{i+1}, \quad x^- \cdot e_i = (m-i+1)e_{i-1}, \quad h \cdot e_i = (2i-m)e_i$$

the action of J(h) (resp.  $J(x^{\pm})$ ) being a times that of h (resp.  $x^{\pm}$ ). To make contact with the theory of highest weight representations, we need:

PROPOSITION 2.6. The action of the generators  $h_k, x_k^{\pm}$  on  $W_m(a)$  is given by:

(1) 
$$x_k^+ \cdot e_i = \left(a - \frac{1}{2}m + i + \frac{1}{2}\right)^k (i+1)e_{i+1};$$

(2) 
$$x_k^- \cdot e_i = \left(a - \frac{1}{2}m + i - \frac{1}{2}\right)^k (m - i + 1)e_{i-1}$$
;

(3) 
$$h_k \cdot e_i = \left\{ \left( a - \frac{1}{2}m + i - \frac{1}{2} \right)^k i(m - i + 1) \right\}$$

$$-\left(a-\frac{1}{2}m+i+\frac{1}{2}\right)^{k}(i+1)(m-i)\right\} e_{i}.$$

*Proof.* It is straightforward to check, using the relations (1)-(3) in Theorem 1.2, that these formulas do define a representation of Y. It therefore suffices to check that they also give the correct action of the generators h, J(h),  $x^{\pm}$ ,  $J(x^{\pm})$ . This is another straightforward computation, using the isomorphism  $\phi$  in (1.2).

COROLLARY 2.7. (a)  $W_m(a)$  is a highest weight representation with highest weight  $\mathbf{d} = (d_0, d_1, ...)$  given by

$$d_k = m\left(a + \frac{1}{2}m - \frac{1}{2}\right)^k.$$

(b) The monic polynomial P associated to  $W_m(a)$  is given by

$$P(u) = \left(u - a + \frac{1}{2}m - \frac{1}{2}\right) \left(u - a + \frac{1}{2}m - \frac{3}{2}\right) \dots \left(u - a - \frac{1}{2}m + \frac{1}{2}\right).$$

*Proof.* (a) It is clear that  $e_m$  is a highest weight vector for  $W_m(a)$  relative to Y. The eigenvalues of the  $h_k$  on  $e_m$  are as stated.

(b) By Theorem 2.4(b), the polynomial P is determined by

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} m \left( a + \frac{1}{2}m - \frac{1}{2} \right)^k u^{-k-1}$$

$$= \frac{\left(u - a + \frac{1}{2}m + \frac{1}{2}\right)}{\left(u - a - \frac{1}{2}m + \frac{1}{2}\right)}.$$

The stated P clearly satisfies this equation.

In section 4 we shall need to consider the duals of the evaluation representations  $W_m(a)$ . If V is any finite-dimensional representation of Y, its dual  $V^*$  is naturally a representation of  $Y^{op}$ , the vector space Y with the opposite multiplication:

$$x.y(in\ Y^{op}) = y.x(in\ Y)\ .$$

Moreover,  $Y^{op}$  is a Hopf algebra with the same co-multiplication as Y.

PROPOSITION 2.8. There is an isomorphism of Hopf algebras  $\theta: Y \to Y^{op}$  such that

$$\theta(x) = -x$$
,  $\theta(J(x)) = J(x)$ 

for all  $x \in \mathfrak{Sl}_2$ .

*Proof.* It is sufficient to prove that the assignment  $x \mapsto -x$ ,  $J(x) \mapsto J(x)$  extends to a homomorphism of Hopf algebras  $Y \to Y^{op}$ . The relations in  $Y^{op}$  are obtained by inserting a minus sign on the right-hand side of relations (1) and (3) in (1.1). The result is now clear.

Remark. The anti-homomorphism  $\theta: Y \to Y$  is closely related to the antipode S of Y, which is given by

$$S(x) = -x$$
,  $S(J(x)) = -J(x) + \frac{1}{4}cx$ ,

where c is the eigenvalue of the Casimir operator in the adjoint representation of  $\mathfrak{sl}_2$  (which depends of course on the choice of inner product (,) on  $\mathfrak{sl}_2$ ).

Thus, if V is a finite-dimensional representation of Y, then  $V^*$  is a representation of Y with action

$$(y. f) (v) = f(\theta(y).v),$$

for  $y \in Y$ ,  $v \in V$  and  $f \in V^*$ . Moreover, the fact that  $\theta$  preserves the comultiplication implies that  $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$  for any two representations  $V_1$ ,  $V_2$  of Y.

COROLLARY 2.9. As representations of Y, we have

$$W_m(a)^* \cong W_m(-a)$$
.

*Proof.* On  $W_m(a)$ , J(x) acts as ax. Therefore, on  $W_m(a)^*$ , J(x) acts as -ax.

The following is a related result.

PROPOSITION 2.10. Every evaluation representation  $W_m(a)$  has a non-degenerate invariant symmetric bilinear form.

This means that there is a non-degenerate symmetric bilinear form < , > on  $W_m(a)$  such that

$$(2.11) \langle y.v_1, v_2 \rangle = \langle v_1, \omega(y).v_2 \rangle$$

for all  $y \in Y$ ,  $v_1, v_2 \in W_m(a)$ .

*Proof.* It is well-known that the representation  $W_m$  of  $\mathfrak{Sl}_2$  carries a form <, > which satisfies (2.11) for all  $y \in \mathfrak{Sl}_2$ . Moreover, the form is unique up to a scalar multiple because  $W_m$  is irreducible. To prove (2.11) in general, it suffices to check the case  $y = x_k^+$ , since the case  $y = x_k^-$  then follows because <, > is symmetric, and  $\omega(x_k^+) = x_k^-$ . Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$(2.12) \langle x_k^+ . e_i, e_{i+k} \rangle = \langle e_i, x_k^- . e_{i+k} \rangle$$

(with the understanding that  $e_i = 0$  unless  $0 \le i \le n$ ). This follows easily from Proposition 2.6 and the invariance of <, > under  $\mathfrak{gl}_2$ .

# 3. A COMBINATORIAL INTERLUDE

The form of the polynomial P associated to the representation  $W_m(a)$  in Corollary 2.7(b) suggests the following definition.

Definition 3.1. A non-empty finite set of complex numbers is said to be a string if it is of the form  $\{a, a + 1, ..., a + n\}$  for some  $a \in \mathbb{C}$  and some  $n \in \mathbb{N}$ .

The centre of the string is  $a + \frac{n}{2}$  and its length is n + 1.

We shall also need:

Definition 3.2. Two strings  $S_1$  and  $S_2$  are said to be non-interacting if either