Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CONTACT GEOMETRY AND WAVE PROPAGATION

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Kapitel: §3. Submanifolds

DOI: https://doi.org/10.5169/seals-57908

Let us consider a hypersurface in a contact manifold, which has no characteristic points. Such a hypersurface is foliated (locally fibrated) into characteristics. The set of characteristics is (at least locally) a manifold whose dimension is two less than the dimension of the initial contact manifold.

THEOREM. *The manifold of characteristics inherits a contact structure from the initial contact manifold.*

A formal way of proving this theorem is a direct calculation. According to the preceding formulae, the characteristics of the hypersurface $K = 0$ are the orbits of the vector field $W = V - K\partial/\partial z$, where $L_V\alpha = K_z\alpha$. Since $i_{\partial/\partial z}d\alpha = 0$, $i_{\partial/\partial z}\alpha = 1$, we have

$$L_W\alpha = K_z\alpha - dK\,.$$

The second term vanishes along the hypersurface $K = 0$. Thus the flow of the vector field W on the hypersurface $K = 0$ preserves the field of hyperplanes $\alpha = 0$ and hence defines a field of hyperplanes on the space of orbits of this field.

Another way of proving this theorem is to consider just one particular case, say the hypersurface, defined in Darboux coordinates by the equation $p_1 = 0$. In this case the characteristic direction is $\partial/\partial q_1$. Hence the space of the characteristics is the coordinate space with Darboux coordinates (z, p, q) where $\tilde{p} = (p_2, \dots, q_n)$ $\tilde{q} = (q_2, \dots, q_n)$. Thus the form $\alpha = dz + \dfrac{pdq + qdp}{2}$ induces on the manifold of characteristics of the hypersurface $p_1 = 0$ the form $\tilde{\alpha} = dz + \dfrac{\tilde{p}d\tilde{q} - \tilde{q}d\tilde{p}}{2}$, as was required.

Now the general case can be reduced to this particular case, since all the hypersurfaces in a contact manifold are locally contactomorphic in neighbourhoods of their non-characteristic points, which follows from the general theorem of Givental, described below.

§3. SUBMANIFOLDS

The submanifolds of a Euclidean or a Riemannian manifold have interior and exterior geometries. For instance, the Gaussian curvature belongs to the interior geometry of the Riemannian metric on the submanifold, while the mean curvature depends on its exterior geometry. In both symplectic and contact geometries the situation is simpler: the local exterior geometry is reduced to the interior one.

THEOREM (Givental). *A submanifold of a contact manifold is locally defined (up to a contactomorphism) by the restriction of the contact structure to the submanifold.*

The contact structure is here considered locally as the module of differential 1-forms vanishing at the contact hyperplanes (over the algebra of functions). The restriction is the module of the restrictions of these forms to the submanifold (over the algebra of functions on the submanifold). Geometrically this means that we consider the field of intersections of the contact planes with the tangent planes of the submanifold, taking the multiplicities into account.

The above theorem follows from a similar theorem on contact forms. We call a contact form transversal to a submanifold if the characteristic directions of the form are nowhere tangent to the submanifolds.

THEOREM (Givental). *Let us consider a homotopy* α_t *of 1-forms defining (different) contact structures on a manifold. Suppose that they are all transversal to a given submanifold, and that they coincide on all the tangent vectors to the submanifold. Then there exists a diffeomorphism of a neighbourhood of the submanifold, identical on the submanifold and transforming* α_0 *to* α_1.

Proof. By a standard homotopy argument the problem is reduced to the solution of the "homology equation"

$$L_{V_t}\alpha_t = \beta_t, \quad \beta_t = -\partial\alpha_t/\partial t,$$

where V_t is the unknown vectorfield, vanishing on our submanifold M and β_t is a known 1-form, vanishing on the tangent vectors to M.

Since the smooth dependence of V on t is easily attainable in the construction that follows, we omit from now on the subscript t to simplify the notations.

Let $V = U + W$ be the decomposition of the (unknown) vector field into its horizontal and vertical parts:

$$i_U\alpha = 0, \quad i_W d\alpha = 0.$$

Let W_0 be a vertical vector for which $i_{W_0}\alpha = 1$. Then $W = fW_0, f = i_W\alpha$. By the homotopy formula $L_V = i_V d + di_V$ we obtain

$$L_V\alpha = i_U d\alpha + df.$$

Thus the homology equation has the form

$$i_U d\alpha + df = \beta$$

where β vanishes at the tangent vectors of M and both the unknown function f and the unknown horizontal field U should vanish on M.

Let π be the geodesic projection of the tubular neighbourhood of M onto M and p be the operator, associating to any k-chain c in this neighbourhood the $k+1$ chain pc formed by the shortest paths of its points x to πx. Then $\partial(pc) + p(\partial c) = c - \pi c$. Let $p^! := \Omega^k \to \Omega^{k-1}$ be the dual of p on the differential forms defined by the identity $\int_{pc} \omega = \int_c p^! \omega$. It is clear that $p^! \omega$ is a differential form, vanishing at the points of M on every k-vector of the ambient space. By duality

$$p^! d\omega + dp^! \omega = \omega - \pi^* \omega \ .$$

For $\omega = \beta$, $\pi^*\beta = 0$ since β vanishes on the tangent vectors of M. Hence $\beta = p^! d\beta + dg$, $g = p^!\beta$. The function g vanishes on M and the form $p^! d\beta$ vanishes at the points of M on every vector of the ambient space. Now the homological equation takes the form

$$i_U d\alpha + df = p^! d\beta + dg \ ,$$

where the unknown f and U should vanish at M. Evaluating the left and the right hand sides at the vertical vector W_0, we find

$$i_{W_0} df = h$$

where $h = i_{W_0} p^! d\beta + \dot{L}_{W_0} dg$ is a known function. This equation is easily solved for f (one integrates along the verticals). The solution f can be chosen to vanish on M (even on any hypersurface, containing M and transversal to the verticals, i.e. to the characteristics of the form α). After choosing f we obtain for U the residual homological equation

$$i_U d\alpha = \gamma \, ,$$

where the 1-form $\gamma = p^! d\beta + d(g-f)$ is zero at the vertical: $i_{W_0}\gamma = 0$, by the choice of f. The condition $i_{W_0}\gamma = 0$ implies the solvability of the equation $i_U d\alpha = \gamma$ for U. The horizontal solution $U(i_U\alpha = 0)$ is unique, since the form $d\alpha$ is nondegenerate at the hyperplane $\alpha = 0$. Hence the horizontal solution U is smooth.

Now we use the ambiguity of the choice of f to reduce U to zero at M. The function f is defined uniquely adding a function which is constant along the characteristics and which vanishes at M. At the points of M the value $i_{W_0}d(g-f)$ is zero (since $p^! d\beta$ vanishes on M). Hence $d(g-f)$ coincides at the points of M with the differential of a function k, constant along the verticals ($k=g-f$ at some hypersurface, transversal to the verticals and containing M).

By adding k to f we do not change the value of df at W_0 and of f at M. But we change γ: now $\gamma = p^! d\beta$ at some hypersurface, containing M and transversal to W_0. Hence $\gamma = 0$ on every vector of the ambient space at the points of M. Thus now $U = 0$ on M, and hence $V = 0$ on M, as required.

The theorem on the contact structures is an easy corollary of this theorem on contact forms.

COROLLARY 1. *The Darboux theorem.*

Proof. $M =$ (point).

COROLLARY 2. *All the Legendre manifolds of any given dimension are locally equal (contactomorphic).*

Proof. All zeroes are equal.

COROLLARY 3. *All hypersurfaces in a contact manifold are contactomorphic in some neighbourhoods of any of its non-characteristic points.*

Proof. The restrictions of the contact forms are equivalent, since they are induced from the natural contact form on the spaces of the characteristics of the hypersurfaces, and those contact forms are equivalent by the Darboux theorem.

Thus we obtain the normal forms for the maximally non-degenerate 1-forms in the even-dimensional spaces: $\alpha = dz - ydx$ for a space of dimension $2n+2$ with coordinates $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, z, w.

The simplest degenerations of the differential 1-form on a manifold are classified by J. Martinet [Ma]:

$$(1+p_1)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n)$$

$$\pm\, dz^2 + (1+p_1)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n+1)$$

$$(1\pm p_1^2)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n)\,.$$

Comparing with the Givental theorem, we obtain the

COROLLARY 1. *For a generic even dimensional submanifold of a contact space the nongeneric points form a set of codimension 2, and in a neighbourhood of the generic points the contact structure is maximally nondegenerate (reducible to the form, $dz = ydx$ for some coordinates x, y, z, w, $\dim\{x\} = 1$).*

COROLLARY 2. *A generic odd dimensional submanifold in a contact manifold inherits a contact structure at its generic points. At the points of some hypersurface the restriction of the contact structure to the manifold is reducible to one of the two (nonequivalent) forms $\pm dz^2 = (1+p_1)dq_1 + p_2dq_2 + \dots + p_kdq_k$.*

Remark. The above classification of the submanifolds depends on the classification of the contact structures (= modules of forms) and not on the forms' classification.

A differential 1-form in a neighbourhood of its nonzero point is either locally equivalent to one of the Darboux or Martinet normal forms, discussed above, or this form is not finitely-determined (is not determined by any finite segment of its Taylor series up to a diffeomorphism). The simplest example of such a nonfinitely determined form is the form $(1+y^3+xy)dy$ on the plane. The codimension of the corresponding event is two.

At present the classification of the degeneration of contact structures (not forms) has acquired the same level of sophistication as the other problems in singularity theory. In the works of M. Zhitomirskii the list of first degenerations is calculated, including all the simple singularities (a singularity is simple, if it has a neighbourhood intersecting a finite number of classes of equivalence). These results of Zhitomirskii, taking into account the Givental theorem, describe also the submanifolds in the contact space up to a local diffeomorphism.

Unfortunately, in most applications one needs the classification of nonsmooth subvarieties of the contact space, for instance, that of unions of intersecting submanifolds.

Example. Let us consider a hypersurface in a Riemannian space. The description of this situation in terms of the contact geometry implies the analysis of a pair of hypersurfaces in the contact space (the symplectic variant of this theory is developed by Sato, Oshiva and Melrose under the name of the theory of glancing rays).

Let us denote our closed Riemannian manifold by M, and the given hypersurface by ∂M.

We shall start from the contact manifold $J^1(M, \mathbf{R})$ of 1-jets of functions on M. Let us consider two hypersurfaces in this space:

$$SJ^1(M, \mathbf{R}) \to J^1(M, \mathbf{R}) \leftarrow \partial J^1(M, \mathbf{R}) .$$

The left hypersurface is defined by the Hamilton-Jacobi equation $p^2 = 1$. It is the contact equivalent of the Riemannian metric. The right hypersurface is formed by the jets of functions on M at the points of ∂M. It is the contact equivalent of the hypersurface ∂M in M.

We shall see that a large part of the Riemannian geometry of the hypersurface ∂M in M may be formulated in terms of these two hypersurfaces in the contact space. Since the contact geometry of these two hypersurfaces is (more or less) independent of their origin, we can apply the knowledge of Riemannian geometry and even the intuition of Euclidean space to the general case of an arbitrary pair of hypersurfaces Y, Z in a contact space X. Let us first consider this general situation.

The hypersurfaces Y and Z intersect generically along a submanifold W of codimension two in X (we suppose that the intersection is transversal). So we obtain the diagram of inclusions

$$\begin{tikzcd}
& X^{2n+1} & \\
Y^{2n} \arrow[ur] & & Z^{2n} \arrow[ul] \\
& W^{2n-1} \arrow[ul] \arrow[ur] \arrow[uu, hook] &
\end{tikzcd}$$

We shall also suppose that the hypersurfaces Y and Z are not tangent to the contact planes (that condition is generically satisfied at a neighbourhood of W since the characteristic points of the hypersurfaces Y and Z are generically isolated).

Hence each of the two hypersurfaces is foliated into its characteristics. Locally (and sometimes globally) this foliation is a fibration, that is there exists a space of characteristics (the base of the fibration). Let us denote the fibrations into characteristics by $Y^{2n} \to U^{2n-1}$ and $Z^{2n} \to V^{2n-1}$ (strictly speaking, U and V are defined only for the germs of Y and Z at a point of W).

Let us consider the composite mappings

$$(\text{via } Y) U^{2n-1} \leftarrow W^{2n-1} \to V^{2n-1} (\text{via } Z) .$$

These two mappings of manifolds of equal dimensions may have singularities. Let us consider the sets of their singular points (points, where the Jacobian matrix's determinant vanishes).

Under some very mild restrictions the sets of critical points of both mappings coincide. Indeed, let us suppose, that tangent hyperplanes of the hypersurfaces Y and Z and the contact hyperplane $\alpha = 0$ form a generic triple of hyperplanes (at some point 0 of W).

LEMMA. *The characteristic of the hypersurface* Y *is tangent to* Z *at the point* 0 *if and only if the restriction of the contact form of* X^{2n+1} *to* W^{2n-1} *degenerates at* 0.

Proof. Let us denote the intersections of the tangent planes to $X, \ldots, W$ at 0 with the contact hyperplane $\alpha = 0$ by the corresponding lower case letters $x, \ldots, w$. Let ξ be the characteristic vector of Y at 0. If ξ is tangent to Z, it belongs to w. Since ξ is skew-orthogonal to y, ξ is skew-orthogonal to w. Hence $d\alpha$ degenerate at w, as required.

Let $d\alpha$ be degenerate at w. Since $\dim w = 2n - 2$ is even, $\dim \operatorname{Ker}(d\alpha | w)$ is at least 2. Let η be a vector, transversal to w in y. Then the equation $d\alpha(\xi, \eta) = 0$ has nontrivial solutions $\xi \in \operatorname{Ker}(d\alpha | w)$. These solutions ξ are skew-orthogonal to η and to w. Hence they are the characteristic vectors of Y at 0. Thus the characteristic vectors of Y at 0 are tangent to W (and hence to Z), as was required.

The lemma is thus proved. Since the condition on the restriction of α to W in the lemma is symmetrical with respect to Y and Z, the lemma implies

COROLLARY. *The characteristics of the hypersurface* Y *are tangent to* W *at the same points as the characteristics of the hypersurface* Z.

Hence the sets Σ of the critical points of our mappings of W to U and to V coincide. We have thus obtained the following hexagonal commutative diagram of mappings

$$\begin{tikzcd}
 & X^{2n+1} & \\
Y^{2n} \arrow[ur, hook] \arrow[dd] & & Z^{2n} \arrow[ul, hook'] \arrow[dd] \\
 & W^{2n-1} \arrow[uu, hook] \arrow[ul, hook'] \arrow[ur, hook] \arrow[dl] \arrow[dr] & \\
U^{2n-1} & & V^{2n-1} \\
 & \Sigma^{2n-2} \arrow[uu, hook] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where X^{2n+1}, U^{2n-1} and V^{2n-1} are equipped with contact structures, and Σ is the set of degenerescence of the restriction of the contact structure of X to W and at the same time the set of critical points of both mappings of W to U and to V.

The dimensions of the kernels of the derivatives of these mappings can't exceed one, since they are the restrictions of the corank 1 projections $Y^{2n} \twoheadrightarrow U^{2n-1}$ and $Z^{2n} \twoheadrightarrow V^{2n-1}$. Hence for the generic hypersurfaces Y and Z the singularities of the mappings $W \to U$ and $W \to V$ are, up to diffeomorphisms, the standard Whitney singularities.

One may even choose the coordinates in Y and U (or in Z and V) in such a way, that the hypersurface W in Y will be defined locally by the equation

$$y^{k+1} + u_1 y^{k-1} + \dots + u_k = 0$$

and the projection $Y \twoheadrightarrow U$ — by the formula

$$(y; u_1, \dots, u_{2n-1}) \mapsto (u_1, \dots, u_{2n-1}) .$$

Example 1. $k = 1$, $n \geqslant 1$. The mapping $W^{2n-1} \to U^{2n-1}$ has a fold singularity at the surface Σ^{2n-2} where $u_1 = y^2$, $y = 0$.

The characteristics $u =$ const. of Y intersect W twice in the neighbourhood of Σ, defining on W an involution. The hypersurface $\Sigma^{2n-2} \subset W^{2n-1}$ is the set of fixed points of this involution.

Hence at the generic points of Σ two involutions $W \to W$ are defined: one interchanges the two points of intersection of W with the characteristics of Y, the other — with the characteristics of Z. Both these involutions have the same hypersurface Σ of fixed points.

Example 2. $k = 2$, $n \geqslant 2$. The mapping $W^{2n-1} \to U^{2n-1}$ has a smooth hypersurface Σ^{2n-2} of critical points, which are the fold points or the cusp singularities. The cusp singularities form a smooth hypersurface $\Sigma^{2n-3} \to \Sigma^{2n-2}$. The set of critical values (the projection of Σ^{2n-2} to U) is a hypersurface in U with a cuspidal edge (projection of Σ^{2n-3}).

Now let us see what is the meaning of all this "general nonsense" in concrete situations.

Example. Let us return to the case of a hypersurface $\partial M : f(q) = 0$ in a Euclidean space $M = \mathbf{R}^n$. In this case $X = J^1(M, \mathbf{R})$, $Y : p^2/2 - 1/2 = 0$ is the Hamilton-Jacobi equation, $Z : f(q) = 0$ defines the hypersurface. The hexagonal diagram takes the form

$$\begin{tikzcd}
 & & J^1(M, \mathbf{R}) & & \\
SJ^1(M, \mathbf{R}) \arrow[urr, hook] \arrow[dd, two heads] & & & & \partial J^1(M, \mathbf{R}) \arrow[ull, hook'] \arrow[dd, two heads] \\
 & & W \arrow[uu, hook] \arrow[ull, hook'] \arrow[urr, hook] \arrow[dll, "I"'] \arrow[drr] & & \\
ST^*M & & & & J^1(\partial M, \mathbf{R}) \\
 & & SJ^1(\partial M, \mathbf{R}) \arrow[uu] \arrow[ull, "\exp"] \arrow[urr] & &
\end{tikzcd}$$

Comments. The characteristics of the Hamilton-Jacobi hypersurface $SJ^1(M, \mathbf{R})$ are the orbits of the geodesic flow in the space ST^*M of the spherical cotangent bundle, equipped with a parameter (the "value" of the jet), increasing along the geodesic with a velocity equal to one.

Fixing the value of this parameter, say $t = 0$, we obtain a point of the characteristic, that is a (cotangent) vector of length one at some point of M, equipped with the 0 "value". Thus we identify the space of characteristics with the space of the spherical cotangent bundle ST^*M (this identification *depends* on the choice $t = 0$).

The projection I associates to a point of ∂M, together with a vector on M of length 1 at that point and a "value" t, the unit tangent vector (on the same line as the original vector) based at a point at a distance t (in the backward direction) from the original point.

A characteristic of $\partial J^1(M, \mathbf{R})$ consists of the 1-jets of all the extensions of a fixed function on ∂M to M at some fixed point of ∂M. The manifold of characteristics is naturally identified with the manifold $J^1(\partial M, \mathbf{R})$ of the 1-jets of functions on ∂M, equipped with its natural contact structure.

The projection II: $W \to J^1(\partial M, \mathbf{R})$ associates to a 1-jet of a function on M, having gradient of length one, the 1-jet of its restriction to ∂M. This projection has the fold singularities on the surface Σ, formed by the 1-jets of the functions on M, whose gradients are of lengths one and are tangential to ∂M. The projection II maps the hypersurface Σ diffeomorphically to the set of critical values of this projection. This set of critical values consists of the 1-jets of functions on ∂M with gradients of length one. Hence we may identify Σ with $SJ^1(\partial M, \mathbf{R})$.

The mapping exp: $SJ^1(\partial M, \mathbf{R}) \to ST^*M$ associates to a 1-jet of a function on ∂M, whose gradient has length one and is tangent to ∂M, a vector of length one on the same straight line as the given vector, but based at the point at a distance t (in the backward direction).

The singularities of the mapping I represent the "inflections" of ∂M.

Example. Let $n = 2$, that is ∂M is a generic plane curve. The mapping I has a fold singularity at the point [of W, corresponding to the unit tangent vectors of ∂M] where the curvature of ∂M is nonzero, and a cusp singularity at the [points of W corresponding to the] inflection points.

Let $n = 3$, that is ∂M is a generic surface in $\mathbf{R}^3$. The mapping I has folds at the points of W corresponding to the generic unitary tangent vectors, cusps at the vectors of asymptotic directions, the swallowtail singularity at the biasymptotic vectors (where the order of tangency of the surface with the tangent line is 3, which is higher than for an ordinary asymptotic vector). The biasymptotical directions exist on a generic surface along a curve; at some special points of this curve there exist triasymptotic directions, the order of tangency is 4 and in Whitney normal form for the singularity of the mapping I we have $k = 4$.

Thus the geometry of a hypersurface in a Euclidean (or in a Riemannian) space, when translated into the microlocal language of contact geometry, leads to the problem of classification (up to contactomorphism) of hypersurfaces with special singularities: of the unions of two smooth and transversal hypersurfaces (Y and Z) in a contact manifold X.

The simplest case ($k = 1$) was studied by Melrose. The normal form of the pair in Darboux coordinates is

$$q_1 = 0, \quad q_1 = p_1^2 + p_2 .$$

This is a formal (or C^∞?) normal form of a generic pair of hypersurfaces in a contact space. In the analytical case the normallizing series are, as a rule, divergent. In the 3-dimensional contact space the normal form of the pair is ($z = q, p^2 = q$) [Me]).

For further results on normal forms in the contact geometry of tangencies see [A3], [La] and [A6].

The state of art in this domain is at present far from the final death of the subject: in most cases the results are know only at the formal level of power series which are usually divergent in the analytical case.

Example 1. Let us consider the product of a swallowtail surface with a Euclidean space, defined in an N-space with coordinates $A, B, \ldots$ by the equation: $\exists t: x^4 + Ax^2 + Bx + C = (x+t)^2 \ldots \forall x$. A generic symplectic structure in a neighbourhood of the origin is formally reducible to the normal form

$$dA \wedge dD + dC \wedge dB + dE \wedge dF + \ldots$$

by a swallowtail preserving diffeomorphic; a generic contact structure — to the Landis normal form

$$\alpha = dZ - DdA - CdB - EdF - \ldots .$$

These normal forms serve probably in the C^∞ case too, but this is not proved.

Example 2. Let us consider a quadratic cone surface in a $2n + 1$ contact space with coordinates $A, B, \ldots$ given by $A^2 + B^2 = C^2$.

The local reduction of such a surface to a normal form by a contactomorphism is important for the study of the transformations of waves, defined by linear hyperbolical systems, derived from variational principles (see [A8] and [A9]).

The formal normal forms of the hypersurfaces with conical singularities in the Darboux coordinates ($\alpha = dz + \frac{pdq - qdp}{2}$) are

$$p^2 \pm q^2 = z^2 + cz^3 \quad (n=1), \qquad p_1^2 \pm q_1^2 = q_2^2 \quad (n>1).$$

These normal forms describe an interior transformation of waves of one kind (say "longitudinal") into waves of other kinds (say, "transverse") in inhomogeneous anisotropic media. The corresponding effect in homogeneous media is the Hamilton conical refraction. In the nonhomogeneous case the geometry of rays is different.

Let us consider the case $n = 1$, that is wave propagations for space-time of dimension 2. The preceding normal form describes two families of characteristics in space-time tangent at one point.

The characteristics through this point are formed from the branches of two smooth (analytic) curves, tangent at that point but having different curvatures. Let those curves be 12 and 34, then the first family's singular characteristic is 14, and that of the second — 32.

The contact of the two characteristics at the origin produces some singular scattering of the family of characteristics of the first (second) type, which is smooth (analytic) outside the origin. Let us consider a nonsingular characteristic of the first family, starting at a small distance ε from the point 1 (where the distances of points 1-4 to the origin are of order 1). The endpoint of this characteristic, taken at the level of point 4, lies at some distance from the singular characteristic 14, namely, at a distance $a\varepsilon + b\varepsilon^2 ln\varepsilon + \dots$. The logarithmic term describes the scattering at the origin: in a regular family the distance would be $a\varepsilon + b\varepsilon^2 + \dots$.

In the 3-dimensional physical space (i.e. for space-time of dimension 4) generic wave fronts (travelling in inhomogeneous media and governed by a variational principle) acquire singular lines, connecting them with waves of different kinds and moving with the wave fronts.

It is interesting to note that the case $n = 1$ is more difficult than $n > 1$. The results are at present formal in both cases. They probably hold for the C^∞ problem both for $n = 1$ and $n > 1$. The divergence of the normalizing series in the analytical problem is proven in the case $n = 1$, while for $n > 1$ there exists still some hope that the series converges. The qualitative results, described above, are independent of the convergence of the series: we need only finite segments of the series.

§4. Legendre fibrations and singularities

The simplest examples of Legendre fibrations are the projectivized cotangent bundles

$$PT^*V^n \to V^n$$

and the "forgetting of derivatives" mappings

$$J^1(M, \mathbf{R}) \to J^0(M, \mathbf{R})$$

(in coordinates: $(p, q, y) \mapsto (q, y)$).

Definition. A *Legendre fibration* is a fibration of a contact manifold with Legendre fibres.

THEOREM. *All the Legendre fibrations of a given dimension are locally contactomorphic (locally = in a neighbourhood of any point of the total space).*

To prove this theorem it is sufficient to construct a local isomorphism of an arbitrary Legendre fibration with one of the preceding examples.