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$$c(v) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where } T = \frac{1}{2} \operatorname{sgn}(\kappa') (Q - i)$$

and Q is as in §6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§8. RELATION WITH CARTAN'S MOVING FRAMES

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succinctly in [7]. The canonical line bundle

$$p: \xi \rightarrow \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} & & \\ \downarrow & p \downarrow & \\ z = \frac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) & & \end{array}$$

Let $\sigma: (\alpha, \beta) \rightarrow \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda \hat{\sigma}$, $f_2 = \dot{f}_1 = \lambda(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\dot{} = \frac{d}{dt}$. Thus (f_1, f_2) is a frame in \mathbf{C}^2 . We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 &= \text{Area}(f_1, f_2) = \text{Area}(\lambda(z_1(t), z_2(t)), \lambda(\dot{z}_1, \dot{z}_2)) \\ &= \lambda^2(z_1 \dot{z}_2 - z_2 \dot{z}_1), \quad \text{or } 1 = -(\lambda z_2)^2 \dot{z} \end{aligned}$$

Thus $\lambda = \frac{i}{z_2 \sqrt{\dot{z}}}$ will do, and we have

$$\begin{aligned} f_1 &= \frac{i}{\sqrt{\dot{z}}} (z, 1), \\ \text{and } f_2 &= \dot{f}_1 = -\frac{1}{2} i \ddot{z} \dot{z}^{-3/2} (z, 1) + i \dot{z}^{-1/2} (z, 0). \end{aligned}$$

Finally a calculation shows that $\dot{f}_2 = S f_1$, where $S = \frac{3}{4} \ddot{z}^2 \dot{z}^{-2} - \frac{1}{2} \ddot{z} \dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ . Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use

inversive arc-length as the parameter, so that the “curvature” S becomes an *intrinsic* invariant of the curve in inversive geometry. And in this case it turns out that S has constant imaginary part. To see this we describe S in terms of the more familiar Euclidean curvature and its derivatives with respect to Euclidean arc-length.

The Euclidean and inversive arc-lengths are related by the equation:

$$dv = |\kappa'|^{1/2} ds, \quad \text{where } ' = \frac{d}{ds}. \text{ Thus:}$$

$$\dot{z} = z' |\kappa'|^{-1/2} = e^{i\theta} |\kappa'|^{-1/2},$$

$$\ddot{z} = \text{sgn}(\kappa') e^{i\theta} \left\{ -\frac{1}{2} \frac{\kappa''}{\kappa'^2} + i \frac{\kappa}{\kappa'} \right\},$$

$$\ddot{\dot{z}} = \text{sgn}(\kappa') e^{i\theta} |\kappa'|^{-1/2} \left\{ -\frac{1}{2} \frac{\kappa'''}{\kappa'^2} + \frac{\kappa''^2}{\kappa'^3} - \frac{\kappa^2}{\kappa'} + i \left(1 - \frac{3}{2} \frac{\kappa \kappa''}{\kappa'^2} \right) \right\}$$

Using these expressions we can calculate the Schwartzian as:

$$\frac{3}{4} \left(\frac{\ddot{\dot{z}}}{\dot{z}} \right)^2 - \frac{1}{2} \frac{\ddot{\dot{z}}}{\dot{z}} = \text{sgn}(\kappa') \left\{ \frac{4(\kappa''' - \kappa^2 \kappa') \kappa' - 5\kappa''^2}{16\kappa'^3} - \frac{i}{2} \right\} = \frac{1}{2} \text{sgn}(\kappa') (Q - i)$$

Regarding the vectors f_1 and f_2 as column vectors, we obtain a 2 by 2 matrix $h = (f_2, f_1) \in G$, and according to the calculation above we have:

$$(f_2, f_1) = (f_2, f_1) \begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}$$

Thus $h(v)$ and $g(v)$ (cf. §7) are equal up to left multiplication by a constant element of G . This interprets Cartan’s canonical frame (f_1, f_2) as the unique frame (up to a constant element of G) forming the columns of a matrix in G which moves the standard curve $y = x^3/6$ to the given curve with contact up to 4th order at the given point.

§9. LOXODROMES

To calculate the curves with Q constant we solve the equation:

$$\frac{dg}{dv} = g \begin{pmatrix} 0 & 1 \\ \frac{1}{2} \varepsilon(Q - i) & 0 \end{pmatrix}, \quad \text{where } \varepsilon = \text{sgn}(\kappa')$$