

§7. The canonical map $g:G$

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The stabilizer of the expression $y = x^4/24 + O(x^6)$ has order 2 and is generated by

$$z \mapsto -\bar{z}.$$

It follows that the non-degeneracy of a vertex is an invariant of inversive geometry.

§7. THE CANONICAL MAP $g: \gamma \rightarrow G$

The considerations of the last section allow us to define a canonical map $g_\gamma: \gamma \rightarrow G$ for vertex free curves γ by mapping a point $p \in \gamma$ to $g_\gamma(p) \in G$, which is the unique group element such that $g_\gamma(p)^{-1}$ sends p to the origin and $g_\gamma(p)^{-1}(\gamma)$ has oriented contact of order 4 with the standard curve $y = x^3/6$ at the origin. We note that if $\gamma' = h(\gamma)$ for some $h \in G$, then obviously $g_{\gamma'}(h(p)) = h(g_\gamma(p))$. Of course altering the initial choice of the origin and the axes used there to describe the model will alter g_γ , but only by right multiplication by some fixed element of G . If $\sigma: (\alpha, \beta) \rightarrow \mathbf{C}$ is a parametrization of the curve by Euclidean arc-length s , and $\sigma'(s) = e^{i\theta(s)}$, then the curvature of the curve at $\sigma(s)$ is $\theta'(s) = \kappa(s)$, and we have the following explicit formula for g .

$$g(s) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\kappa'' - 2i\kappa\kappa')/4\kappa' & 1 \end{pmatrix} \begin{pmatrix} |\kappa'|^{-1/4} & 0 \\ 0 & |\kappa'|^{1/4} \end{pmatrix}$$

The first two factors are Euclidean motions whose inverse puts γ into oriented first order contact with the oriented x -axis. The rest improve the order of contact to 4 as in §6. It is convenient to regard g as a function of the inverse arc-length v . Now $g(v)$ is a curve on the Lie group G , with tangent vector dg/dv at $g(v)$. Left translation by $g(v)^{-1}$ moves this tangent vector to the origin to yield

$$(7.1) \quad c(v) = g(v)^{-1} \frac{dg}{dv}$$

which is a vector in the Lie algebra $sl_2(\mathbf{C})$ of 2 by 2 complex matrices of trace zero. As v varies $c(v)$ inscribes a curve on this Lie algebra. Indeed it is well known (e.g. [13], p. 71) that this curve determines the original curve $g(v)$ up to left translation by an arbitrary constant element of G . Here is an explicit formula for the curve $c(v)$. It is easy but rather tedious to verify it.

$$c(v) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where } T = \frac{1}{2} \operatorname{sgn}(\kappa') (Q - i)$$

and Q is as in §6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§8. RELATION WITH CARTAN'S MOVING FRAMES

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succinctly in [7]. The canonical line bundle

$$p: \xi \rightarrow \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{c} (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} \\ \downarrow \quad p \downarrow \\ z = \frac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) \end{array}$$

Let $\sigma: (\alpha, \beta) \rightarrow \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda \hat{\sigma}$, $f_2 = \dot{f}_1 = \lambda(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\dot{} = \frac{d}{dt}$. Thus (f_1, f_2) is a frame in \mathbf{C}^2 . We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 &= \text{Area}(f_1, f_2) = \text{Area}(\lambda(z_1(t), z_2(t)), \lambda(\dot{z}_1, \dot{z}_2)) \\ &= \lambda^2(z_1 \dot{z}_2 - z_2 \dot{z}_1), \quad \text{or } 1 = -(\lambda z_2)^2 \dot{z} \end{aligned}$$

Thus $\lambda = \frac{i}{z_2 \sqrt{\dot{z}}}$ will do, and we have

$$\begin{aligned} f_1 &= \frac{i}{\sqrt{\dot{z}}} (z, 1), \\ \text{and } f_2 &= \dot{f}_1 = -\frac{1}{2} i \ddot{z} \dot{z}^{-3/2} (z, 1) + i \dot{z}^{-1/2} (z, 0). \end{aligned}$$

Finally a calculation shows that $\dot{f}_2 = S f_1$, where $S = \frac{3}{4} \ddot{z}^2 \dot{z}^{-2} - \frac{1}{2} \ddot{z} \dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ . Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use