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Combining this with 4.5 yields

$$\mathbf{n}(\tau(f)) - \tau(\mathbf{n}(f)) = [\mathbf{n}, \tau](f) = \kappa_1 \tau(f) + \mu_1 \mathbf{n}(f) = \kappa_1 \tau(f) + \mathbf{n}\tau(f)$$

which gives 4.6 as required.

§5. A GENERALIZED FOUR VERTEX THEOREM

The curves of constant curvature in the round 2-sphere S^2, the upper half plane H^2 (hyperbolic space), and the Euclidean plane $\mathbf{R}^2$ are just the circles. Moreover, the stereographic projection $p: S^2 \to \mathbf{R}^2$ and the inclusion $i: H^2 \to \mathbf{R}^2$ both preserve these circles. Thus theorem 4.2 says that our form

$$\omega = \sqrt{|\kappa'(s)|}\, ds$$

along a curve γ in $\mathbf{R}^2$ pulls back via p or i to the form

$$\omega = \sqrt{|\kappa_g'(s)|}\, ds$$

along the corresponding curve γ', where here $\kappa_g(s)$ and s refer to the geodesic curvature and arc-length of γ' in the metric for S^2 or H^2. Thus we obtain the four vertex theorem for S^2 and H^2. It follows that the four vertex theorem holds for all complete simply connected Riemannian surfaces of constant curvature. Finally if γ is a null-homotopic smooth simple closed curve on an arbitrary complete Riemannian surface M of constant curvature, then γ lifts one-to-one to a smooth simple closed curve with the same number of vertices on the simply connected universal cover of M. Once again it follows that the number of vertices is at least four.

Remark 5.1. Interestingly, simple closed homotopically non-trivial curves in the real projective plane always have at least three vertices [17]. Note that in non-orientable surfaces the number of honest vertices of a closed curve need not necessarily be even, since here geodesic curvature is only defined up to a sign.

§6. NORMAL FORM AND INVERSIVE CURVATURE

Let p be a non-vertex point of an oriented curve γ. Since the subgroup of Euclidean motions in G acts transitively on the points of $\mathbf{R}^2$ and the unit tangent vectors at these points, we may assume that the point $p \in \gamma$ which