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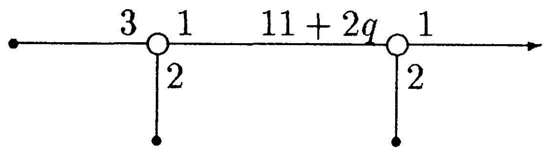
diagrams are not equivalent. This is the counterexample to the spectrum conjecture found by Steenbrink and Stevens, cf. [SSS].

6. Equations

In this section we discuss the equations of series: what do we have to add to f to obtain a required element of its series?

In the example $W^\#$ at the beginning of section 4, we had:

$$W^\#_{1,2q-1} : (y^2 - x^3)^2 + x^{4+q}y$$



The Puiseux expansion of $W^\#_{1,\infty}$: $f(x, y) = (y^2 - x^3)^2$, is $x = t^2, y = t^3$. When we substitute this in $x^{4+q}y$, we get t^{11+2q}, which is just the number N in the EN-diagram.

More generally, it appears that adding $\varphi \in \mathcal{O}$ with $\varphi(t^2, t^3)$ of order $11 + 2q$, gives the same result, although there are various kinds of exceptions.

In theorem 6.5 below, we give conditions on φ such that $f + \varepsilon\varphi$ has the required type, where ε is introduced in order to fulfil transversality properties. This avoids exceptional cases such as when $f(x, y) = y^2$ and $\varphi(x, y) = 2x^k y + x^{2k}$, the sum is then a non-isolated singularity.

Again, f has only transversal A_1 singularities; but the following lemma is valid in greater generality.

6.1. Lemma. *Let $f, \psi \in \mathcal{O}$ and assume f has a non-isolated singularity. If for all small $\varepsilon > 0$, $f + \varepsilon\psi$ has a singularity topologically equivalent to f, then for almost all ε the zero sets of f and $f + \varepsilon\psi$ are equal.*

Proof. First take $f(x, y) = y^n$ with $n > 1$. Assume that for no ε, f and $f + \varepsilon\psi$ have the same zero set. Then we may assume $f + \varepsilon\psi = \big(y + F(x, \varepsilon)\big)^n$ where $F(x, \varepsilon) \not\equiv 0$, regarded also as a function, of ε, can be written as

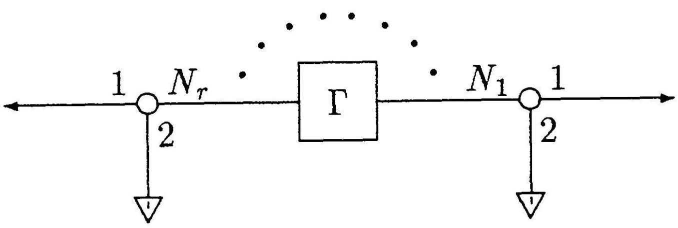
$$F(x, \varepsilon) = \sum_{i>0} a_i(\varepsilon)x^i .$$

Here $a_i(\varepsilon)$ may have positive fractional powers of ε. $f + \varepsilon\psi$ is linear in ε. By writing out the equation

$$\frac{\partial}{\partial \varepsilon}(y + F(x, \varepsilon))^n = 0 ,$$

one immediately sees that this impossible. If f is not of the form y^n then there always is a small neighbourhood away from the origin where it is. There we can apply the above argument. □

6.2. Let $f = f_1^2 \cdots f_r^2 g$ have EN-diagram Γ, and let $N_i > N_{0i}$ and c_i be defined as usual. We are looking for φ with the property that $f + \varepsilon\varphi$ has the topological type of EN-diagram $\Gamma^* = \Gamma^*(N_1, \ldots, N_r)$:

$\Gamma^* =$ 

By Puiseux's Theorem [Ph], we can choose coordinates x, y of $\mathbf{C}^2$ in such a way that the Puiseux expansions of the $\Sigma_i, (1 \leqslant i \leqslant r)$ have the form:

$$\begin{cases} x = t^{n_i} \\ y = \eta_i(t) = \sum_{k \geqslant 1} c_{ik} t^k \end{cases}$$

For each i we have the valuation function $\upsilon_i\colon \mathscr{O} \to \mathbf{N} \cup \{\infty\}$ given by

$$\upsilon_i(\varphi) = \operatorname{ord}_t \varphi(t^{n_i}, \eta_i(t)) = \dim_{\mathbf{C}} \mathscr{O}/(f_i, \varphi) .$$

After considering various examples, one is tempted to think that whenever for all i, $\upsilon_i(\varphi) = N_i + c_i$, $f + \varepsilon\varphi$ has, for general ε, the required topological type given by EN-diagram $\Gamma^*(N_1, \ldots, N_r)$. The following example shows that this is not true. Take $f(x, y) = y^2$, and $\varphi(x, y) = x^k y + x^N$. Although $\upsilon(\varphi) = N$, the topological type is determined by k and not by N when $2k < N$. So we have to take care of low order multiples of f. We will do this by considering υ and an extra valuation $\upsilon^{(2)}$.

6.3. *Definition.* Consider $h = h_{\text{red}}^2$, where $h_{\text{red}} \in \mathscr{O}$ is irreducible with Puiseux expansion $x = t^n, y = \sum a_i t^i$. Let β be the largest characteristic exponent. For $\alpha \in \mathbf{C}$, $n \in \mathbf{N}$, define $w_{\alpha,N}\colon \mathscr{O} \to \mathbf{N} \cup \{\infty\}$ by:

$$w_{\alpha,N}(\varphi) = \operatorname{ord}_\tau\left(\varphi(\tau^{2n}, \sum a_i \tau^{2i} + \alpha\tau^{2\beta + N - N_0})\right) .$$

Finally, define $\upsilon^{(2)}\colon \mathscr{O} \to \mathbf{N} \cup \{\infty\}$ by:

$$v^{(2)}(\varphi) = \begin{cases} \min_{\alpha \neq 0} w_{\alpha, v(\varphi)}(\varphi) & \text{if} \quad v(\varphi) < \infty\,, \\ \min_{\alpha \neq 0} w_{\alpha, 2v(\varphi/h)}(\varphi) & \text{if} \quad v(\varphi) = \infty\,. \end{cases}$$

Notice that $v^{(2)}(\varphi) = \infty \Leftrightarrow \varphi \in (h)$. If $N - N_0$ is odd, the number $v^{(2)}(\varphi)$ is equal to the intersection number of φ with some curve which has as its Puiseux pairs the Puiseux pairs of h_{red} with one extra pair, $(N - N_0, 2)$, added.

6.4. *Example.* Take $f(x, y) = y^2$ and $\varphi(x, y) = x^k y + x^N$. Then $v(\varphi) = N$ and $v^{(2)}(\varphi) = \min\{2k + N, 2N\}$. Observe that the type of f is A_{m-1} with $m = v^{(2)}(\varphi) - v(\varphi)$.

6.5. Let $f = f_1^2 \cdots f_r^2 g$ be as above. For $1 \leqslant i \leqslant r$ we now have valuations $v_i^{(2)}$ as in the preceding definition. Recall $\Gamma^*(N_1, \ldots, N_r)$ is obtained from the EN-diagram Γ of f by replacing all multiple arrows as in the last picture.

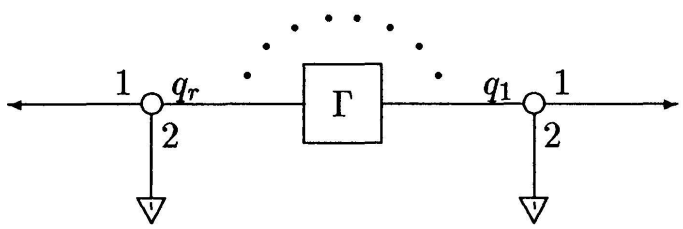
THEOREM. *Suppose* $\varphi \in \mathscr{P}$ *satisfies* $v^{(2)}(\varphi) = v_1^{(2)}(\varphi) \cdots v_r^{(2)}(\varphi) < \infty$. *Then* $f + \varepsilon\varphi$ *has, for almost all* $\varepsilon \neq 0$, *the topological type given by EN-diagram* $\Gamma^*(N_1, \ldots, N_r)$, *with for* $1 \leqslant i \leqslant r$:

(a) $N_i = v_i^{(2)}(\varphi) - v_i(\varphi) - c_i$ *if* $v_i(\varphi) < \infty$, *or*

(b) $N_i = 2v_i(\varphi/f_i) - c_i$ *if* $v_i(\varphi) = \infty$,

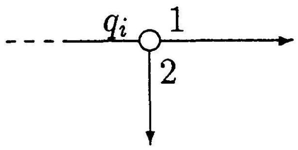
provided that $N_i > N_{i0}$ *for all* i.

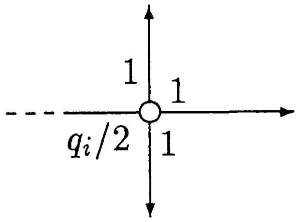
Proof. Since the order of $\varphi|\Sigma_i$ is $> N_{0i} + c_i$ and ε is general (use lemma 6.1), the good minimal resolution of $f + \varepsilon\varphi$ also resolves the singularities of f. So the EN-diagram of f is a subdiagram of that of $f + \varepsilon\varphi$. Hence, according to theorem 3.4, $f + \varepsilon\varphi$ has the EN-diagram:

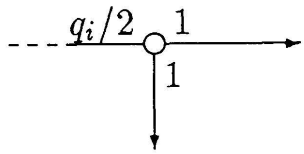


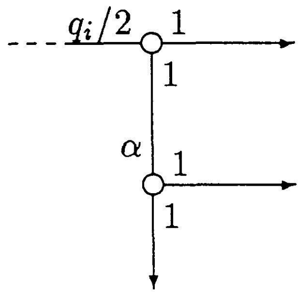
for certain numbers q_i, $(1 \leqslant i \leqslant r)$. It remains to prove that q_i equals the number N_i stated in the theorem.

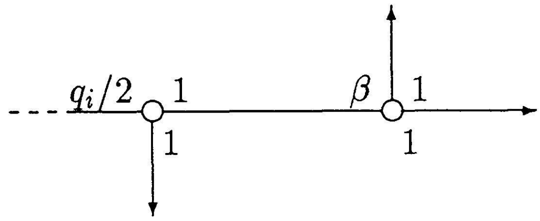
For this purpose we consider one specific i at the time, and draw f_i in the same picture as $f + \varepsilon\varphi$. That is, we draw the EN-diagram of their product, unless f_i happens to be a branch of $f + \varepsilon\varphi$. Using an argument analogous to the one presented in section 3.3 (which provided the two possible extensions to the EN-diagram), we conclude that the situation near f_i is as in one of the five following cases:











In each picture, the arrow pointing downwards represents f_i. Observe that when f_i is removed (replaced by a dot) we get back the situation of the original picture as it should. We now compute q_i in each case, using the interpretation of the valuations as intersection numbers with f_i. Recall that they can be computed by walking from arrow to arrow in the EN-diagram, see [EN], section 10. To clarify matters, we explain in each case the local situation as follows. In the resolution of f we take suitable local coordinates u, v near the strict transform of the branch f_i in such a way that $f_i^2 = v^2$ and that the branches of $f + \varepsilon\varphi$ near f_i have the form mentioned.

Pictures # 1, # 2 and # 5: One computes $v_i(\varphi) = q_i + c_i$ and $v_i^{(2)} = 2q_i + 2c_i$. Therefore $q_i = N_i$. In picture # 1, q_i is odd and in picture # 2 even. In both cases the local situation is $v(v^2 + u^s)$ with $s = q_i - N_{i0}$. In picture # 5, the two branches have intersection number $\beta > q_i/2$ with each other.

Picture # 3: One computes $v_i(\varphi) = \infty$ and $v_i(\varphi/f_i) = q_i/2 + c_i$. Therefore $q_i = N_i$. The local situation is $v(v + u^{s/2})$ with s as before.

Picture # 4: One computes $v_i(\varphi) = q_i/2 + \alpha + c_i$ and $v_i^{(2)}(\varphi) = 3q_i/2 + \alpha + 2c_i$. Again we obtain that q_i equals the number N_i of the theorem. The local situation is $v(v^2 + u^{s/2}v + u^{\alpha + s/2})$. □

6.6. *Remark.* We want to point out at this point that it is easy to find a φ satisfying the condition. One can use the method of [EN], pages 57-58. An interesting observation is, that in general the monomials of φ themselves will have a smaller order in t than φ.

6.7. THE CASE THAT f IS ARBITRARY.

If $f = f_1^{m_1} \cdots f_r^{m_r} g$ with f_i irreducible, $m_i \geqslant 2$ and g reduced, we still have that $f + \varepsilon\varphi$ has the diagram of f with the multiple arrows replaced. We know exactly which replacements are possible (see section 3.8). To find out what is the type of $f + \varepsilon\varphi$, it again suffices to investigate linking behaviour. Some possibilities that only become apparent when f_i and $f + \varepsilon\varphi$ are drawn in one diagram (that is the diagram of their product), have to be opted out by considering linking with cables which are known to be correct, using such valuations as $\upsilon^{(2)}$.

Although the tests become increasingly difficult, this gives a way to generalize theorem 6.5.

6.8. IOMDIN TYPE SERIES.

We end with a remark on series of the form $f + \varepsilon l^k$, where l is a linear form not tangent to any branch of f and $k \geqslant k_0$, the largest polar ratio of f. These series have been studied by Iomdin and Lê, see [Lê], not only in the curve case but for general dimensions. Siersma [Si] has given a formula for the Δ_* of these series. In the curve case this is just a special case of our results. Notice that:

$$\upsilon_i(l) = d_i k \quad \text{where} \quad d_i = e_l(\Sigma_i) = \Sigma_i \cdot l\,,$$
$$\upsilon_i^{(2)}(l) = 2 d_i k\,.$$

We would like to stress again that these Iomdin type series are generally much coarser than our topological series: they are single indexed and for example the Milnor number increases with steps of $d = d_1 + \cdots + d_r$ within the series.

APPENDIX

In this appendix the EN-diagrams of the series of plane curve singularities listed in [AGV] are drawn.

The first part consists of the *exceptional families E, W* and *Z*.

The second part contains the *infinite series A, D, J, W, $W^{\#}$, X, Y* and *Z*. All variants are given. In the tables, we have that: