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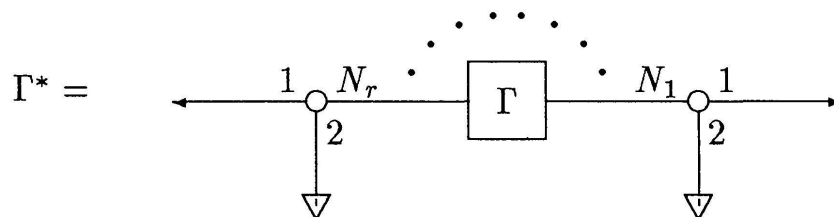
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Proof. If $q = 1$ the theorem is clear. Now suppose $q > 1$. Consider the behaviour of the polynomials Δ_0, Δ^1 and Δ' under this splice operation. Splicing introduces a new edge E which contributes to Δ^1 with a factor $t^q - 1$. This introduces new 2×2 -Jordan blocks. Both splice components have $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right)$ in their spectrum (coming from Δ'). But, as both eigenvalues in a 2×2 -block are of different weight, L has $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right) + \left(\frac{i}{q} \right)$ instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of L' and L'' have to be added. \square

5. INVARIANTS IN THE CASE THAT f HAS ONLY TRANSVERSAL A_1 SINGULARITIES

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal A_1 singularities.

Throughout this section, $f \in \mathcal{S}$ is of the form $f = f_1^2 \cdots f_r^2 g$, with f_1, \dots, f_r irreducible and g reduced. The critical set of f is $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, and the transverse type of f along Σ_i is A_1 . For all $i \in \{1, \dots, r\}$, we have numbers N_{0i} and c_i as defined in section 3.3. Let $N_i > N_{0i}$ ($1 \leq i \leq r$). According to theorem 3.4, a typical element of the series belonging to f has the topological type (EN-diagram) Γ^* :



That is: each arrow of the EN-diagram Γ of f belonging to a double component, is replaced in the way described in theorem 3.4. So varying the N_i will give us the complete series belonging to f .

The following two propositions are easy consequences of theorem 3.4. Let $N = (N_1, \dots, N_r)$ and let f_N have topological type Γ^* .

5.1. PROPOSITION. *Let $\Delta_*[f]$ and $\Delta_*[f_N]$ be the Δ_* of f and f_N respectively. Then:*

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^r (t^{N_i + c_i} - (-1)^{N_i}).$$

\square

5.2. PROPOSITION. Let μ_∞ be the Milnor number of f and μ_N that of f_N . Let $\mu_0 \in \{1, 2\}$ be the number of connected components of the Milnor fibre of f . Then:

$$\mu_N = \mu_\infty - \mu_0 + 1 + \sum_{i=1}^r (N_i + c_i).$$

The numbers μ_∞, μ_0 and c_i ($1 \leq i \leq r$) depend only on f . □

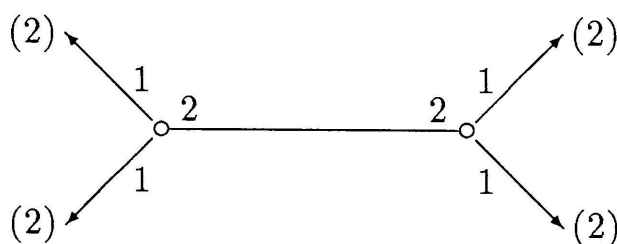
5.3. We conclude this list of topological invariants with the formula of the spectrum of a series (see section 4).

PROPOSITION. Define for $1 \leq i \leq r$: $\gamma_i = 0$ if c_i is even and $\gamma_i = 1/2$ when c_i is odd. Write $v_i = N_i + c_i$. Then:

$$\text{Sp}(f_N) = \text{Sp}(f) + \sum_{i=1}^r \sum_{j=0}^{v_i-1} \left(\frac{1}{2} - \frac{\gamma_i + j}{v_i} \right)$$

Proof. One can use the proof of [St], theorem 4.5, but it is also possible to work out the various cases using the method of section 4. For the proof of [St], the following observation is needed. Let F'_i be a transversal slice transverse to Σ_i (in this case F'_i consists of two points — the transverse type is A_1). Let $T_i: H_0(F'_i) \rightarrow H_0(F'_i)$ be the monodromy of the local system over the punctured disc $\Sigma_i \setminus \{0\}$. Then it is well-known that T_i is the identity if c_i is even and — identity if c_i is odd. In fact, even if the transversal type is not A_1 , the following holds. Let $t_i: H_0(F'_i) \rightarrow H_0(F'_i)$ be the Milnor fibration monodromy of f restricted to a transversal slice through $x \in \Sigma_i$. Then t_i is a cyclic permutation of the finite number of points in F'_i , and $T_i = t_i^{-c_i}$. □

Example. Let $f(x, y) = (y^2 - x^4)^2(x^2 - y^4)^2$; its EN-diagram is:



Observe that according to the proposition, the spectrum of f_N is independent of the order of N_1, \dots, N_4 . If we take $N = (5, 5, 6, 6)$ and $N = (5, 6, 5, 6)$ we get the same spectrum but different topological types, because the EN-

diagrams are not equivalent. This is the counterexample to the spectrum conjecture found by Steenbrink and Stevens, cf. [SSS].

6. EQUATIONS

In this section we discuss the equations of series: what do we have to add to f to obtain a required element of its series?

In the example $W^\#$ at the beginning of section 4, we had:

$$W_{1,2q-1}^\# : (y^2 - x^3)^2 + x^{4+q}y$$

The Puiseux expansion of $W_{1,\infty}^\# : f(x, y) = (y^2 - x^3)^2$, is $x = t^2, y = t^3$. When we substitute this in $x^{4+q}y$, we get t^{11+2q} , which is just the number N in the EN-diagram.

More generally, it appears that adding $\varphi \in \mathcal{O}$ with $\varphi(t^2, t^3)$ of order $11 + 2q$, gives the same result, although there are various kinds of exceptions.

In theorem 6.5 below, we give conditions on φ such that $f + \varepsilon\varphi$ has the required type, where ε is introduced in order to fulfil transversality properties. This avoids exceptional cases such as when $f(x, y) = y^2$ and $\varphi(x, y) = 2x^k y + x^{2k}$, the sum is then a non-isolated singularity.

Again, f has only transversal A_1 singularities; but the following lemma is valid in greater generality.

6.1. LEMMA. *Let $f, \psi \in \mathcal{O}$ and assume f has a non-isolated singularity. If for all small $\varepsilon > 0$, $f + \varepsilon\psi$ has a singularity topologically equivalent to f , then for almost all ε the zero sets of f and $f + \varepsilon\psi$ are equal.*

Proof. First take $f(x, y) = y^n$ with $n > 1$. Assume that for no ε , f and $f + \varepsilon\psi$ have the same zero set. Then we may assume $f + \varepsilon\psi = (y + F(x, \varepsilon))^n$ where $F(x, \varepsilon) \not\equiv 0$, regarded also as a function, of ε , can be written as

$$F(x, \varepsilon) = \sum_{i>0} a_i(\varepsilon)x^i.$$

Here $a_i(\varepsilon)$ may have positive fractional powers of ε . $f + \varepsilon\psi$ is linear in ε . By writing out the equation