We now give a formula giving the number of essentially different diagrams with one node and only multiplicities less than m, that can be spliced to a component of multiplicity m.

PROPOSITION. *The number is:*

$$\sum_{q|m} \mathfrak{p}\,(m/q) + \sum_{1 \leqslant p \leqslant m-1}\ \sum_{q|(m-p),\, q>1} \mathfrak{p}\,((m-p)/q) - 1$$

where $\mathfrak{p}\,(n)$ *is the number of integer partitions of* n.

Proof. In such a diagram at most one dot appears, with at the node a weight $\geqslant 2$. The number of edges emerging from the node must be at least 3. There is at most one weight $\geqslant 1$. These are consequences of the algebraicity condition. The splice condition demands that the total linking number of the other components with the splice component equals m. The formula is now a matter of counting. □

For $m \leqslant 15$ we obtain:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
number	0	2	4	9	12	22	27	42	54	76	91	134	159	211	263

This can be regarded as an upperbound on the number of symbols (such as A, $W^{\#}$, etc.) needed to give names to all singularities of corank m.

4. THE SPECTRUM OF A PLANE CURVE SINGULARITY

4.1. In this section we compute the spectrum of a plane curve singularity from the EN-diagram and we prove a splice formula for spectra. This will be needed in the next section, where we look at several invariants within a series. First we need to define a number of polynomials.

4.2. We denote by F the Milnor fibre of a plane curve singularity f.

Definition.

$$\Delta_0\,(t) = \text{char. pol. of } H_0(h): H_0(F) \to H_0(F)\ ,$$
$$\Delta_1\,(t) = \text{char. pol. of } H_1(h): H_1(F) \to H_1(F)\ ,$$
$$\Delta_*(t) = \Delta_1(t)/\Delta_0(t) \in \mathbf{Q}(t)$$

Recall that $H_0(F)$ and $H_1(F)$ have ranks d and μ, respectively, where d equals the number of connected components and μ the Milnor number.

We will also need the following polynomials. Let $h_*: H_1(F) \to H_1(F)$ be the algebraic monodromy.

Definition:

(a) Δ^1 is the characteristic polynomial of $h_* | \mathrm{Ker}(h_*^N - 1)$, where N is a common multiple of the order of the eigenvalues of h_*,

(b) Δ' is the characteristic polynomial of $h_* | \mathrm{Im}\big(H_1(\partial F) \to H_1(F)\big)$.

The roots of Δ^1 are the eigenvalues of the 2×2-Jordan blocks of h_*.

Observe that all polynomials defined above can be obtained easily from the EN-diagram, cf. [EN], section 11 and [Ne].

4.3. The *spectrum* of a holomorphic function germ is a set of rational numbers with integral multiplicities, denoted as $\sum_{\alpha \in \mathbf{Q}} n_\alpha(\alpha)$ (an element of the free abelian group on $\mathbf{Q}$), which can be regarded as logarithms of the eigenvalues of the algebraic monodromy.

In the isolated singularity case we have that $\Delta_1(t) = \prod_\alpha \big(t - \exp(2\pi i \alpha)\big)^{n_\alpha}$. In the case of plane curve singularities, the spectrum numbers α satisfy $-1 < \alpha < 1$, so for each eigenvalue $\lambda \neq 1$ there are two possible α's with $\lambda = \exp(2\pi i \alpha)$.

4.4. We follow [St] for a brief description of the spectrum. For details we refer to this source. Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be non-zero holomorphic function germ, and denote by F its Milnor fibre. The reduced cohomology groups $H^*(F) = H^*(F; \mathbf{C})$ carry a canonical mixed Hodge structure. The semi-simple part T_s of the monodromy acts as an automorphism of this mixed Hodge structure, and in particular it preserves the Hodge filtration $\mathscr{F}$. Write $\mathrm{Gr}^p_{\mathscr{F}} = \mathscr{F}^p / \mathscr{F}^{p+1}$, and let s_p be the dimension of $\mathrm{Gr}^p_{\mathscr{F}}$. There are rational numbers α_{pj} with $1 \leqslant j \leqslant s_p$, $n - p - 1 < \alpha_{pj} \leqslant n - p$ such that

$$\det(t \cdot \mathrm{Id} - T_s; \mathrm{Gr}^p_{\mathscr{F}}) = \prod_{j=1}^{s_p} \big(t - \exp(-2\pi i \alpha_{pj})\big)$$

Now we define $\mathrm{Sp}_n\big(H^k(F; \mathbf{C}), \mathscr{F}, T_s\big) = \sum_p \sum_j (\alpha_{pj})$ and:

$$\mathrm{Sp}(f) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{Sp}_n\big(H^k(F), \mathscr{F}, T_s\big)$$

It is clear that the spectrum is a finer invariant than the characteristic polynomial. Steenbrink has proved for instance that the spectrum distinguishes all quasi-homogeneous isolated singularities (not only curves). But already for plane curves the spectrum is not a complete invariant of the topological type. Details of these facts can be found in [SSS].

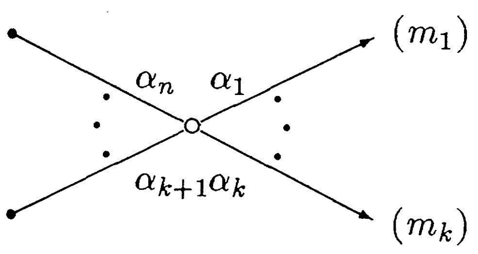
4.5. *Example.* Consider $f(x, y) = xy(y^2 - x^3)$ and $g(x, y) = xy(y - x^5)$. Then f and g have the same integral monodromy (see [MW]), their characteristic polynomial is $\Delta_1 = (t-1)(t^{11}-1)$. But

$$\mathrm{Sp}(f) = \sum_{i \in \{0,1,2,3,4,6\}} \left(-\frac{i}{11}\right) + \left(\frac{i}{11}\right)$$

$$\mathrm{Sp}(g) = \sum_{i \in \{0,1,2,3,4,5\}} \left(-\frac{i}{11}\right) + \left(\frac{i}{11}\right)$$

4.6. In [LS] a method is given to compute the spectrum of a reduced curve singularity from the resolution graph. However, the non-reduced case follows by the same methods. The results are closely related to those of Neumann on the equivariant signatures of the isometric structure on $H_1(F; \mathbf{C})$ given by the monodromy and the sesquilinearized Seifert form, see [Ne]. Below we combine the results of [LS] and [Ne] to obtain a purely topological method to compute the spectrum.

For a root of unity λ the signature σ_λ^- is defined in [Ne] and computed as the sum of the σ_λ^- of all the splice components. Consider a (very general) splice component:



For the moment, put $m_i = 0$ for $i \in \{k+1, \ldots, n\}$; so

$$m = \sum_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n m_j$$

is the multiplicity of the central node. Choose integers $\beta_j (1 \leqslant j \leqslant n)$ with $\beta_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n \equiv 1 \pmod{\alpha_j}$ and put $s_j = (m_j - \beta_j m)/\alpha_j$.

Remark. The numbers s_j are, modulo m, equal to the multiplicities of the neighbour vertices in the resolution graph.

For a real number x, let $\{x\}$ be the fractional part of x, and let

$$((x)) = \begin{cases} \frac{1}{2} - \{x\} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z} \end{cases}$$

4.7. PROPOSITION. *Write* $\lambda = \exp(2\pi i p/q)$ *with* g.c.d. $(p, q) = 1$. *Then we have (see Neumann [Ne]):*

$$\sigma_\lambda^- = \begin{cases} 0 & \textit{if } q \textit{ does not divide } m, \\ 2\sum_{i=1}^n ((s_i p/q)) & \textit{if } q \textit{ divides } m. \end{cases}$$ □

4.8. For λ a root of unity, let $b_{0,\lambda}$, b_λ, b_λ^1, b_λ' be the multiplicities of λ as a root of Δ_0, Δ_1, Δ^1, Δ', respectively (these polynomials have been defined in section 4.2) Let σ_λ^- be the signature as computed above. Write $e(\alpha) = \exp(2\pi i \alpha)$. Sp$(f)$ denotes the spectrum of f.

THEOREM. Sp$(f) = \sum n_\alpha(\alpha)$ *with:*

$$n_\alpha = \begin{cases} (b_{e(\alpha)} + b'_{e(\alpha)} - \sigma^-_{e(\alpha)})/2 & \textit{if } -1 < \alpha < 0 \\ r - 1 \ (r = \#\ \textit{branches}) & \textit{if } \alpha = 0 \\ (b_{e(\alpha)} - b'_{e(\alpha)} + \sigma^-_{e(\alpha)})/2 - b_{0,e(\alpha)} & \textit{if } 0 < \alpha < 1 \end{cases}$$

Proof. The proposition is a translation of the results of [LS], extended to the case of non-reduced singularities. The difference with [LS] is, that the roots of Δ', coming from the boundary, must be added to the weight one part, and the roots of Δ_0 must be subtracted from the weight zero part. In the language of [Ne]: The Γ_λ and the $-\Lambda_\lambda^1$ part contribute to the negative (weight 1) spectrum numbers, the Λ_λ^1 part contributes to the positive (weight 0) spectrum numbers. The pairs of eigenvalues in the 2×2-Jordan blocks are evenly distributed among the positive and negative parts. The roots of Δ_0 give only weight 0 spectrum numbers and they have negative multiplicity. □

4.9. A point which may cause confusion is the fact that in the definition of spectrum *reduced* (co)homology is used. Therefore we define $\mathrm{Sp}_*(f) = \mathrm{Sp}(f) - (0)$. It is now possible to compare Sp_* with Δ_*: If $\mathrm{Sp}_*(f) = \sum_\alpha n_\alpha(\alpha)$, then $\Delta_*(t) = \prod_{\alpha \in \mathbf{Q}} (t - e(\alpha))^{n_\alpha}$.

Example. The A_∞ singularity has $\mathrm{Sp}_* = -\left(\frac{1}{2}\right) - (0)$. Recall that its Δ_* equals $(t^2 - 1)^{-1}$. D_∞ has spectrum Sp $= (0)$, so $\mathrm{Sp}_* = 0$ ('empty'). Let $f(x, y) = (y^2 - x^3)(y^3 - x^2)$ be the A'Campo singularity. Then:

$$\mathrm{Sp}_*(f) = \left(-\frac{1}{2}\right) + 2\left(-\frac{3}{10}\right) + 2\left(-\frac{1}{10}\right) + 2\left(\frac{1}{10}\right) + 2\left(\frac{3}{10}\right) + \left(\frac{1}{2}\right).$$

As with all isolated singularities, this spectrum is symmetrical (i.e. if (α) is in the spectrum, then so is $(-\alpha)$). This is not the case with non-isolated singularities. The asymmetry comes from the fact that the Milnor fibre can have more than one connected component and from the fact that the monodromy possibly acts non-trivially on the boundary of F. Both can be seen in:

$$\mathrm{Sp}_*(x^2y^2) = \left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right).$$

Observe that the Δ_* of x^2y^2 is just 1, as with D_∞.

4.10. The Δ_* behaves well under splicing: it is the product of the Δ_* of the splice components. Our topological way of looking at spectra asks for a formula of splicing spectra. It appears that $\mathrm{Sp}_* = \mathrm{Sp} - (0)$ is *almost* additive.

Example. In the example above we computed the spectrum of the A'Campo singularity. Both splice components are isomorphic to that of the non-isolated singularity $x^2(y^2-x^3)$, which has spectrum:

$$\mathrm{Sp}_* = \left(-\frac{1}{2}\right) + \left(-\frac{3}{10}\right) + \left(-\frac{1}{10}\right) + \left(\frac{1}{10}\right) + \left(\frac{3}{10}\right).$$

So we have to add both spectra, but instead of $2\left(-\frac{1}{2}\right)$ we have $\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)$. This is the result of the new edge in the EN-diagram, giving a new 2×2-block.

4.11. THEOREM. *Let L be the result of splicing L' and L'' along components S' and S'', respectively. Let $m'(m'')$ be the multilink multiplicity of $S'(S'')$ and put $q = \text{g.c.d.}(m', m'')$. Then*

$$\mathrm{Sp}_*(L) = \mathrm{Sp}_*(L') + \mathrm{Sp}_*(L'') + \sum_{i=1}^{q=1} (i/q) - (-i/q).$$

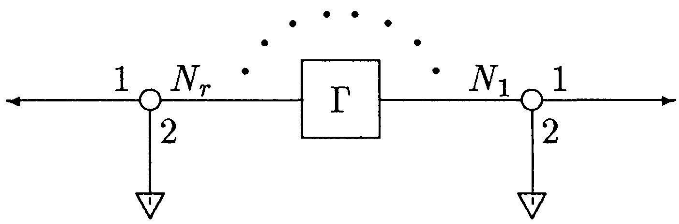
Proof. If $q = 1$ the theorem is clear. Now suppose $q > 1$. Consider the behaviour of the polynomials Δ_0, Δ^1 and Δ' under this splice operation. Splicing introduces a new edge E which contributes to Δ^1 with a factor $t^q - 1$. This introduces new 2×2-Jordan blocks. Both splice components have $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right)$ in their spectrum (coming from Δ'). But, as both eigenvalues in a 2×2-block are of different weight, L has $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right) + \left(\frac{i}{q} \right)$ instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of L' and L'' have to be added. □

5. Invariants in the case that f has only transversal A_1 singularities

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal A_1 singularities.

Throughout this section, $f \in \mathscr{O}$ is of the form $f = f_1^2 \cdots f_r^2 g$, with $f_1, \ldots, f_r$ irreducible and g reduced. The critical set of f is $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, and the transverse type of f along Σ_i is A_1. For all $i \in \{1, \ldots, r\}$, we have numbers N_{0i} and c_i as defined in section 3.3. Let $N_i > N_{0i}$ $(1 \leqslant i \leqslant r)$. According to theorem 3.4, a typical element of the series belonging to f has the topological type (EN-diagram) Γ^*:

$\Gamma^* =$



That is: each arrow of the EN-diagram Γ of f belonging to a double component, is replaced in the way described in theorem 3.4. So varying the N_i will give us the complete series belonging to f.

The following two propositions are easy consequences of theorem 3.4. Let $N = (N_1, \ldots, N_r)$ and let f_N have topological type Γ^*.

5.1. Proposition. *Let* $\Delta_*[f]$ *and* $\Delta_*[f_N]$ *be the* Δ_* *of* f *and* f_N *respectively. Then:*

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^{r} \left(t^{N_i + c_i} - (-1)^{N_i}\right) . \qquad \square$$