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Autor: Schrauwen, Robert
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We now give a formula giving the number of essentially different diagrams with one node and only multiplicities less than m , that can be spliced to a component of multiplicity m .

PROPOSITION. *The number is:*

$$\sum_{q|m} p(m/q) + \sum_{1 \leq p \leq m-1} \sum_{q|(m-p), q > 1} p((m-p)/q) - 1$$

where $p(n)$ is the number of integer partitions of n .

Proof. In such a diagram at most one dot appears, with at the node a weight ≥ 2 . The number of edges emerging from the node must be at least 3. There is at most one weight ≥ 1 . These are consequences of the algebraicity condition. The splice condition demands that the total linking number of the other components with the splice component equals m . The formula is now a matter of counting. \square

For $m \leq 15$ we obtain:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
number	0	2	4	9	12	22	27	42	54	76	91	134	159	211	263

This can be regarded as an upperbound on the number of symbols (such as A , $W^\#$, etc.) needed to give names to all singularities of corank m .

4. THE SPECTRUM OF A PLANE CURVE SINGULARITY

4.1. In this section we compute the spectrum of a plane curve singularity from the EN-diagram and we prove a splice formula for spectra. This will be needed in the next section, where we look at several invariants within a series. First we need to define a number of polynomials.

4.2. We denote by F the Milnor fibre of a plane curve singularity f .

Definition.

$$\Delta_0(t) = \text{char. pol. of } H_0(h): H_0(F) \rightarrow H_0(F),$$

$$\Delta_1(t) = \text{char. pol. of } H_1(h): H_1(F) \rightarrow H_1(F),$$

$$\Delta_*(t) = \Delta_1(t)/\Delta_0(t) \in \mathbf{Q}(t)$$

Recall that $H_0(F)$ and $H_1(F)$ have ranks d and μ , respectively, where d equals the number of connected components and μ the Milnor number.

We will also need the following polynomials. Let $h_*: H_1(F) \rightarrow H_1(F)$ be the algebraic monodromy.

Definition:

- (a) Δ^1 is the characteristic polynomial of $h_* | \text{Ker}(h_*^N - 1)$, where N is a common multiple of the order of the eigenvalues of h_* ,
- (b) Δ' is the characteristic polynomial of $h_* | \text{Im}(H_1(\partial F) \rightarrow H_1(F))$.

The roots of Δ^1 are the eigenvalues of the 2×2 -Jordan blocks of h_* .

Observe that all polynomials defined above can be obtained easily from the EN-diagram, cf. [EN], section 11 and [Ne].

4.3. The *spectrum* of a holomorphic function germ is a set of rational numbers with integral multiplicities, denoted as $\sum_{\alpha \in \mathbf{Q}} n_\alpha(\alpha)$ (an element of the free abelian group on \mathbf{Q}), which can be regarded as logarithms of the eigenvalues of the algebraic monodromy.

In the isolated singularity case we have that $\Delta_1(t) = \prod_{\alpha} (t - \exp(2\pi i\alpha))^{n_\alpha}$. In the case of plane curve singularities, the spectrum numbers α satisfy $-1 < \alpha < 1$, so for each eigenvalue $\lambda \neq 1$ there are two possible α 's with $\lambda = \exp(2\pi i\alpha)$.

4.4. We follow [St] for a brief description of the spectrum. For details we refer to this source. Let $f: (\mathbf{C}^{n+1}, 0) \rightarrow (\mathbf{C}, 0)$ be non-zero holomorphic function germ, and denote by F its Milnor fibre. The reduced cohomology groups $H^*(F) = H^*(F; \mathbf{C})$ carry a canonical mixed Hodge structure. The semi-simple part T_s of the monodromy acts as an automorphism of this mixed Hodge structure, and in particular it preserves the Hodge filtration \mathcal{F} . Write $\text{Gr}^p_{\mathcal{F}} = \mathcal{F}^p / \mathcal{F}^{p+1}$, and let s_p be the dimension of $\text{Gr}^p_{\mathcal{F}}$. There are rational numbers α_{pj} with $1 \leq j \leq s_p$, $n - p - 1 < \alpha_{pj} \leq n - p$ such that

$$\det(t \cdot \text{Id} - T_s; \text{Gr}^p_{\mathcal{F}}) = \prod_{j=1}^{s_p} (t - \exp(-2\pi i\alpha_{pj}))$$

Now we define $\text{Sp}_n(H^k(F; \mathbf{C}), \mathcal{F}, T_s) = \sum_p \sum_j (\alpha_{pj})$ and:

$$\text{Sp}(f) = \sum_{k=0}^n (-1)^{n-k} \text{Sp}_n(H^k(F), \mathcal{F}, T_s)$$

It is clear that the spectrum is a finer invariant than the characteristic polynomial. Steenbrink has proved for instance that the spectrum distinguishes

all quasi-homogeneous isolated singularities (not only curves). But already for plane curves the spectrum is not a complete invariant of the topological type. Details of these facts can be found in [SSS].

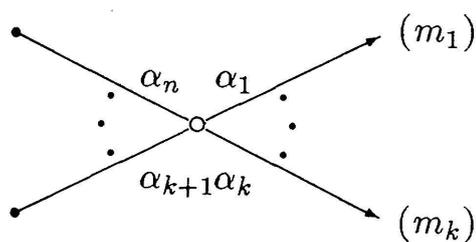
4.5. *Example.* Consider $f(x, y) = xy(y^2 - x^3)$ and $g(x, y) = xy(y - x^5)$. Then f and g have the same integral monodromy (see [MW]), their characteristic polynomial is $\Delta_1 = (t-1)(t^{11}-1)$. But

$$\text{Sp}(f) = \sum_{i \in \{0, 1, 2, 3, 4, 6\}} \left(-\frac{i}{11} \right) + \left(\frac{i}{11} \right)$$

$$\text{Sp}(g) = \sum_{i \in \{0, 1, 2, 3, 4, 5\}} \left(-\frac{i}{11} \right) + \left(\frac{i}{11} \right)$$

4.6. In [LS] a method is given to compute the spectrum of a reduced curve singularity from the resolution graph. However, the non-reduced case follows by the same methods. The results are closely related to those of Neumann on the equivariant signatures of the isometric structure on $H_1(F; \mathbf{C})$ given by the monodromy and the sesquilinearized Seifert form, see [Ne]. Below we combine the results of [LS] and [Ne] to obtain a purely topological method to compute the spectrum.

For a root of unity λ the signature σ_λ^- is defined in [Ne] and computed as the sum of the σ_λ^- of all the splice components. Consider a (very general) splice component:



For the moment, put $m_i = 0$ for $i \in \{k+1, \dots, n\}$; so

$$m = \sum_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n m_j$$

is the multiplicity of the central node. Choose integers β_j ($1 \leq j \leq n$) with $\beta_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n \equiv 1 \pmod{\alpha_j}$ and put $s_j = (m_j - \beta_j m) / \alpha_j$.

Remark. The numbers s_j are, modulo m , equal to the multiplicities of the neighbour vertices in the resolution graph.

For a real number x , let $\{x\}$ be the fractional part of x , and let

$$((x)) = \begin{cases} \frac{1}{2} - \{x\} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z} \end{cases}$$

4.7. PROPOSITION. Write $\lambda = \exp(2\pi ip/q)$ with $\text{g.c.d.}(p, q) = 1$. Then we have (see Neumann [Ne]):

$$\sigma_{\lambda}^{-} = \begin{cases} 0 & \text{if } q \text{ does not divide } m, \\ 2 \sum_{i=1}^n ((s_i p/q)) & \text{if } q \text{ divides } m. \end{cases} \quad \square$$

4.8. For λ a root of unity, let $b_{0,\lambda}, b_{\lambda}, b_{\lambda}^1, b'_{\lambda}$ be the multiplicities of λ as a root of $\Delta_0, \Delta_1, \Delta^1, \Delta'$, respectively (these polynomials have been defined in section 4.2) Let σ_{λ}^{-} be the signature as computed above. Write $e(\alpha) = \exp(2\pi i\alpha)$. $\text{Sp}(f)$ denotes the spectrum of f .

THEOREM. $\text{Sp}(f) = \sum n_{\alpha}(\alpha)$ with:

$$n_{\alpha} = \begin{cases} (b_{e(\alpha)} + b'_{e(\alpha)} - \sigma_{e(\alpha)}^{-})/2 & \text{if } -1 < \alpha < 0 \\ r - 1 \text{ (} r = \# \text{ branches)} & \text{if } \alpha = 0 \\ (b_{e(\alpha)} - b'_{e(\alpha)} + \sigma_{e(\alpha)}^{-})/2 - b_{0,e(\alpha)} & \text{if } 0 < \alpha < 1 \end{cases}$$

Proof. The proposition is a translation of the results of [LS], extended to the case of non-reduced singularities. The difference with [LS] is, that the roots of Δ' , coming from the boundary, must be added to the weight one part, and the roots of Δ_0 must be subtracted from the weight zero part. In the language of [Ne]: The Γ_{λ} and the $-\Lambda_{\lambda}^1$ part contribute to the negative (weight 1) spectrum numbers, the Λ_{λ}^1 part contributes to the positive (weight 0) spectrum numbers. The pairs of eigenvalues in the 2×2 -Jordan blocks are evenly distributed among the positive and negative parts. The roots of Δ_0 give only weight 0 spectrum numbers and they have negative multiplicity. \square

4.9. A point which may cause confusion is the fact that in the definition of spectrum *reduced* (co)homology is used. Therefore we define $\text{Sp}_*(f) = \text{Sp}(f) - (0)$. It is now possible to compare Sp_* with Δ_* : If $\text{Sp}_*(f) = \sum_{\alpha} n_{\alpha}(\alpha)$, then $\Delta_*(t) = \prod_{\alpha \in \mathbf{Q}} (t - e(\alpha))^{n_{\alpha}}$.

Example. The A_{∞} singularity has $\text{Sp}_* = -\left(\frac{1}{2}\right) - (0)$. Recall that its

Δ_* equals $(t^2 - 1)^{-1}$. D_{∞} has spectrum $\text{Sp} = (0)$, so $\text{Sp}_* = 0$ ('empty'). Let $f(x, y) = (y^2 - x^3)(y^3 - x^2)$ be the A'Campo singularity. Then:

$$\begin{aligned} \mathrm{Sp}_*(f) = & \left(-\frac{1}{2}\right) + 2\left(-\frac{3}{10}\right) + 2\left(-\frac{1}{10}\right) + 2\left(\frac{1}{10}\right) \\ & + 2\left(\frac{3}{10}\right) + \left(\frac{1}{2}\right). \end{aligned}$$

As with all isolated singularities, this spectrum is symmetrical (i.e. if (α) is in the spectrum, then so is $(-\alpha)$). This is not the case with non-isolated singularities. The asymmetry comes from the fact that the Milnor fibre can have more than one connected component and from the fact that the monodromy possibly acts non-trivially on the boundary of F . Both can be seen in:

$$\mathrm{Sp}_*(x^2y^2) = \left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right).$$

Observe that the Δ_* of x^2y^2 is just 1, as with D_∞ .

4.10. The Δ_* behaves well under splicing: it is the product of the Δ_* of the splice components. Our topological way of looking at spectra asks for a formula of splicing spectra. It appears that $\mathrm{Sp}_* = \mathrm{Sp} - (0)$ is *almost* additive.

Example. In the example above we computed the spectrum of the A'Campo singularity. Both splice components are isomorphic to that of the non-isolated singularity $x^2(y^2 - x^3)$, which has spectrum:

$$\begin{aligned} \mathrm{Sp}_* = & \left(-\frac{1}{2}\right) + \left(-\frac{3}{10}\right) + \left(-\frac{1}{10}\right) \\ & + \left(\frac{1}{10}\right) + \left(\frac{3}{10}\right). \end{aligned}$$

So we have to add both spectra, but instead of $2\left(-\frac{1}{2}\right)$ we have

$\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)$. This is the result of the new edge in the EN-diagram, giving a new 2×2 -block.

4.11. THEOREM. *Let L be the result of splicing L' and L'' along components S' and S'' , respectively. Let $m'(m'')$ be the multilink multiplicity of $S'(S'')$ and put $q = \mathrm{g.c.d.}(m', m'')$. Then*

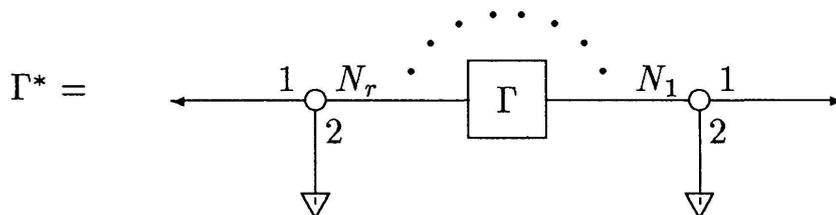
$$\mathrm{Sp}_*(L) = \mathrm{Sp}_*(L') + \mathrm{Sp}_*(L'') + \sum_{i=1}^{q-1} (i/q) - (-i/q).$$

Proof. If $q = 1$ the theorem is clear. Now suppose $q > 1$. Consider the behaviour of the polynomials Δ_0, Δ^1 and Δ' under this splice operation. Splicing introduces a new edge E which contributes to Δ^1 with a factor $t^q - 1$. This introduces new 2×2 -Jordan blocks. Both splice components have $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right)$ in their spectrum (coming from Δ'). But, as both eigenvalues in a 2×2 -block are of different weight, L has $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right) + \left(\frac{i}{q} \right)$ instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of L' and L'' have to be added. \square

5. INVARIANTS IN THE CASE
 THAT f HAS ONLY TRANSVERSAL A_1 SINGULARITIES

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal A_1 singularities.

Throughout this section, $f \in \mathcal{O}$ is of the form $f = f_1^2 \cdots f_r^2 g$, with f_1, \dots, f_r irreducible and g reduced. The critical set of f is $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, and the transverse type of f along Σ_i is A_1 . For all $i \in \{1, \dots, r\}$, we have numbers N_{0i} and c_i as defined in section 3.3. Let $N_i > N_{0i}$ ($1 \leq i \leq r$). According to theorem 3.4, a typical element of the series belonging to f has the topological type (EN-diagram) Γ^* :



That is: each arrow of the EN-diagram Γ of f belonging to a double component, is replaced in the way described in theorem 3.4. So varying the N_i will give us the complete series belonging to f .

The following two propositions are easy consequences of theorem 3.4. Let $N = (N_1, \dots, N_r)$ and let f_N have topological type Γ^* .

5.1. PROPOSITION. Let $\Delta_*[f]$ and $\Delta_*[f_N]$ be the Δ_* of f and f_N respectively. Then:

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^r (t^{N_i+c_i} - (-1)^{N_i}). \quad \square$$