1.2. Signature and oriented skeins

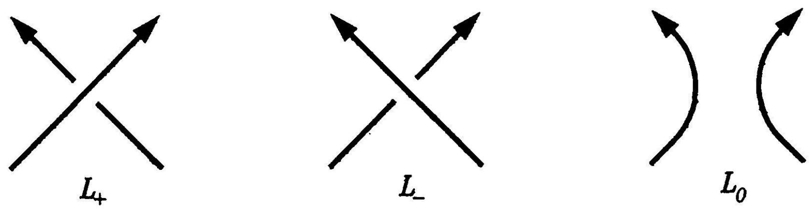

FIGURE 1

There are two equivalence relations with which we may be interested: Let R_b be the equivalence relation on the set of links generated by:

(6) if (L_+, L_-, L_0) and (L'_+, L'_-, L'_0) are skein triples then
$$\begin{cases} L_+R_bL'_+,\ L_-R_bL'_- \Rightarrow L_0R_bL'_0 \\ L_+R_bL'_+,\ L_0R_bL'_0 \Rightarrow L_-R_bL'_- \\ L_-R_bL'_-,\ L_0R_bL'_0 \Rightarrow L_+R_bL'_+ . \end{cases}$$

I shall call this equivalence relation *broad oriented skein equivalence.* The other relation on the set of oriented links, *narrow oriented skein equivalence*, is the equivalence relation R_n generated by:

(7) if (L_+, L_-, L_0) and (L'_+, L'_-, L'_0) are skein triples then
$$\begin{cases} L_+R_nL'_+,\ L_0R_nL'_0 \Rightarrow L_-R_nL'_- \\ L_-R_nL'_-,\ L_0R_nL'_0 \Rightarrow L_+R_nL'_+ \end{cases}.$$

It is obvious that R_b is a weaker equivalence relation than R_n (i.e. the equivalence classes are larger), but it is not clear (and I do not know) whether it is strictly weaker. By the *broad* or *narrow oriented skein of links* I refer to the set of equivalence classes of oriented links under the relation R_b or R_n (Note that in most of the literature, R_n is referred to as "skein equivalence"; R_b is not referred to at all). The polynomial invariant $P_L(l, m)$ of [15], [3] etc. may be regarded as the most general linear broad skein invariant (see [15], [16]). The fact that the value of $P_L(l, m)$ specified on the unknot U is sufficient to define its value on any link may be taken as saying that the broad oriented skein is *generated* by U. The corresponding statement for the narrow oriented skein is that specifying the values of $P_L(l, m)$ on all unlinks is sufficient to define its values on all links — the set of unlinks generates the narrow oriented skein.

1.2. SIGNATURE AND ORIENTED SKEINS

I now show that the signature function $\sigma_L(\xi)$ of any link with non-zero Alexander polynomial is a broad oriented skein invariant (It is already known that the signature $\sigma = \frac{1}{2}\sigma_L(-1)$ is a narrow skein oriented invariant for knots — which are, of course a proper subset of the set of links with non-zero Alexander polynomial — see e.g. [15]). Before proceeding with this, however, I introduce some convenient notation. Given links L and L', set

$$\rho_0(L, L') = \begin{cases} \infty & \text{if } L \neq L' \\ 0 & \text{if } L = L' \end{cases}. \tag{8}$$

Now for $n > 0$ set

$$\rho_{n+1}(L, L') = \min \begin{cases} \min\limits_{\substack{\text{skein triples} \\ (L, L_1, L_2) \\ (L', L_1', L_2')}} (1 + \max\{\rho_n(L_1, L_1'), \rho_n(L_2, L_2')\}), \\ \min\limits_{\substack{\text{skein triples} \\ (L_1, L, L_2) \\ (L_1', L', L_2')}} (1 + \max\{\rho_n(L_1, L_1'), \rho_n(L_2, L_2')\}), \\ \min\limits_{\substack{\text{skein triples} \\ (L_1, L_2, L) \\ (L_1', L_2', L)}} (1 + \max\{\rho_n(L_1, L_1'), \rho_n(L_2, L_2')\}), \\ \min\limits_{\text{Links } L''} (\rho_n(L, L'') + \rho_n(L', L'')). \end{cases} \tag{9}$$

Finally, define

$$\rho(L, L') = \lim_{n \to \infty} \rho_n(L, L') \tag{10}$$

to be the *broad skein distance* from L to L'. It is easy to see that ρ is a metric on the set of links, and that $\rho(L, L') < \infty$ if and only if L and L' are broadly skein equivalent. Intuitively, ρ measures the number of skein triples in a minimal chain of simple skein equivalences needed to establish skein equivalence. It is useful because it provides a grip on broad skein equivalence for use in inductive proofs.

By modifying equation (9) (i.e. by removing the third line inside the outer minimum) a similar metric can easily be defined for narrow skein equivalence, but I shall not need it here.

Note that since skein-equivalent links have the same $P_L(l, m)$ and hence $\Delta_L(t)$ and determinant, it makes sense to speak of the determinant of a skein equivalence class.

THEOREM 1. *The signature is well-defined on broad oriented skein equivalence classes with non-zero determinant.*

Proof. The proof proceeds by induction on the broad oriented skein distance ρ between two links. If $\rho(L, L') = 0$ then $L = L'$ and trivially $\sigma(L) = \sigma(L')$. For the inductive hypothesis suppose that $\rho(L, L') < n$ implies that $\sigma(L) = \sigma(L')$ whenever $\det(L) = \det(L') \neq 0$. I will prove that this is enough to show that the same is true whenever $\rho(L, L') = n$ and $\det(L) = \det(L') \neq 0$. This will suffice to prove the theorem.

Now if $\rho(L, L') = n$ then from the construction of ρ we can see that $\rho_n(L, L') = n$ and $\rho_{n-1}(L, L') = \infty$. One of the following cases must occur, corresponding to the four lines in the outer minimum of equation 9:

1) There exist skein triplets (L, L_1, L_2) and (L', L'_1, L'_2) such that $\rho(L_1, L'_1) \leqslant n - 1$ and $\rho(L_2, L'_2) \leqslant n - 1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$, provided $\det(L_1) \neq 0$ and $\det(L_2) \neq 0$ respectively. Let L, L_1, L_2 be given Seifert surfaces M_+, M_-, M_0 which are identical away from the crossing at which the links differ and near it are as shown in Figure 2.

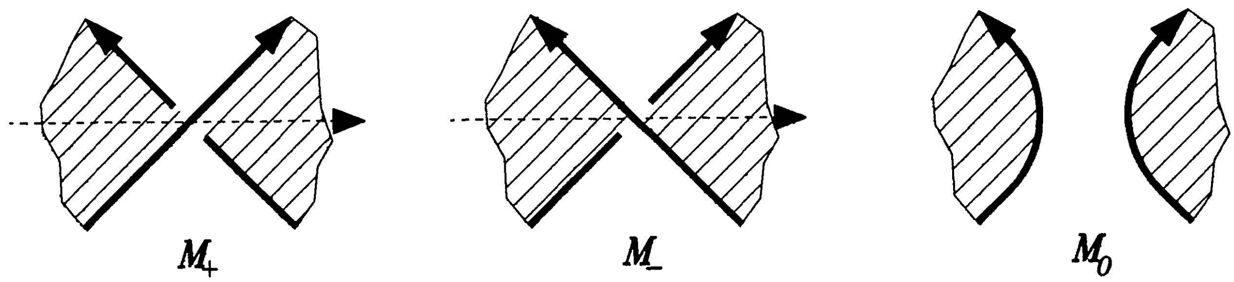

FIGURE 2

Choose a set of generators for $H_1(M_0; Z)$ and extend to sets of generators for the first homologies of M_+ and M_- by including loops which intersect the additional crossing once in the link projection as shown, and which are identical away from this crossing. Then using these to obtain Seifert matrices V_+, V_-, V_0 for the three links, we find that $V_+ + V'_+$, $V_- + V'_-$, $V_0 + V'_0$ are of the form

$$(11) \qquad \begin{pmatrix} 2r & \rho \\ \rho' & S_0 \end{pmatrix}, \begin{pmatrix} 2r+2 & \rho \\ \rho' & S_0 \end{pmatrix}, (S_0).$$

Now if $\det(L_2) \neq 0$, transformations of the form $Q \to PQP'$ (P nonsingular and with rational entries) suffice to put these matrices in the form

$$(12) \qquad \begin{pmatrix} m & 0 \\ 0 & S_0 \end{pmatrix}, \begin{pmatrix} m-1 & 0 \\ 0 & S_0 \end{pmatrix}, (S_0),$$

where $\det(S_0) \neq 0$. Then

$$\begin{aligned}\sigma(L) &= \sigma(V_+ + V'_+)\\ &= \sigma(S_0) + \operatorname{sgn}(m)\\ &= \sigma(L_2) + \operatorname{sgn}\left(\frac{\det(L)}{i\det(L_2)}\right)\\ &= \sigma(L'_2) + \operatorname{sgn}\left(\frac{\det(L')}{i\det(L'_2)}\right)\\ &= \sigma(L')\end{aligned}$$

using the inductive hypothesis for the equality of the third and fourth lines. If, however, $\det(L_2) = 0$, then $\det(L) \neq 0 \Rightarrow \det(L_1) \neq 0$ and the three matrices can be put in the form

$$(14)\qquad \begin{pmatrix} m & 0 & \varepsilon \\ 0 & A & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} m-1 & 0 & \varepsilon \\ 0 & A & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

with $\det(A) \neq 0$ and $\varepsilon = \pm 1$, whence it is easy to check that

$$(15)\qquad \sigma(L) = \sigma(A) = \sigma(L_1) = \sigma(L'_1) = \sigma(L').$$

2) There exist skein triplets (L_1, L, L_2) and (L'_1, L', L'_2) such that $\rho(L_1, L'_1) \leqslant n-1$ and $\rho(L_2, L'_2) \leqslant n-1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$. We may prove by precisely the same arguments as in Case 1) that $\sigma(L) = \sigma(L')$.

3) There exist skein triplets (L_1, L_2, L) and (L'_1, L'_2, L') such that $\rho(L_1, L'_1) \leqslant n-1$ and $\rho(L_2, L'_2) \leqslant n-1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$, provided $\det(L_1) \neq 0$ and $\det(L_2) \neq 0$ respectively. In this case $L = L_0$ in the skein triplet under construction. As above, we may choose Seifert matrices V_+, V_-, V_0 for L_1, L_2, L such that (after transformations of the form $Q \to PQP'$, P non-singular and with rational entries) $V_+ + V'_+, V_- + V'_-, V_0 + V'_0$ take the forms

$$(16)\qquad \begin{pmatrix} m & 0 \\ 0 & S_0 \end{pmatrix}, \begin{pmatrix} m-1 & 0 \\ 0 & S_0 \end{pmatrix}, (S_0),$$

where $\det(S_0) \neq 0$. Then at least one of L_1, L_2 has non-zero determinant. Without loss of generality, suppose $\det(L_1) \neq 0$. Then

$$
\begin{aligned}
\sigma(L) &= \sigma(S_0) \\
&= \sigma(L_1) - \operatorname{sgn}(m) \\
&= \sigma(L_1) - \operatorname{sgn}\left(\frac{\det(L_1)}{i\det(L)}\right) \qquad (17)\\
&= \sigma(L'_1) - \operatorname{sgn}\left(\frac{\det(L'_1)}{i\det(L')}\right) \\
&= \sigma(L')
\end{aligned}
$$

using the inductive hypothesis to establish the equality of the third and fourth lines.

4) There exists a link L'' such that $\rho(L, L'') < n$ and $\rho(L'', L') < n$. Then $\sigma_{L'} = \sigma_{L''} = \sigma_L$.

This concludes the proof of Theorem (1). Note the use made of the fact that it is not possible for exactly one member of a skein triplet to have non-zero determinant. Note also that although the transformations $Q \to PQP'$ (P nonsingular with rational entries) may alter the determinant, they do not change its sign, which is all that the proof requires. □

Unfortunately it is not clear how to deal with the cases in which $\det(L) = 0$, because it is conceivable that one or more of the skein triplets in the chain establishing skein equivalence of two links could have all three determinants equal to zero, in which case the methods of the above proof would not be applicable. It becomes clearer what is going on and that these exceptions are not just artifacts of a poor proof when the more general situation of the signature function $\sigma_L : S^1 \to Z$ is considered.

Now since the Alexander polynomial $\Delta_L(t)$ can be obtained from $P_L(l, m)$ it is clearly a broad oriented skein invariant and it makes sense to state

THEOREM 2. *Broadly skein-equivalent oriented links have the same signature function* $\sigma_L(\omega)$ *for all* ω *other than roots of the Alexander polynomial.*

Proof. The proof of Theorem 4 goes through virtually unmodified ($\Delta_L(\omega)$ takes the place of the determinant, and the obvious changes are made to accommodate the fact that we are dealing with Hermitian matrices instead of symmetric ones). □

Now if we adopt the usual convention (see [5]) that the value of $\sigma_L(\omega)$ at a root of the Alexander polynomial is defined to be the mean of its two "adjacent" values

$$\lim_{\varepsilon \to 0+} \sigma_L(\omega e^{i\varepsilon}) \quad \text{and} \quad \lim_{\varepsilon \to 0-} \sigma_L(\omega e^{i\varepsilon}), \tag{18}$$

the fact that both of these values are well-defined broad oriented skein invariants completes the proof that

COROLLARY 3. *The signature function* $\sigma_L : S^1 \to Z$ *is a broad oriented skein invariant for all links with non-zero Alexander polynomials.* □

This is an intriguing result, especially in view of the fact that $\sigma_L(\omega)$ is known to be a concordance invariant. It is natural to ask what relations there may be between skein theory and concordance theory. Another obvious question is that of what happens when the Alexander polynomial Δ_L is identically zero. In these circumstances the first Alexander ideal of the link collapses and the signature function can be thought of as extracting information about higher Alexander ideals. Kanenobu ([8] and [9]) has shown that there exist infinitely many links with identical P-polynomials but distinct second Alexander ideals, so there is no obvious reason to suppose that this information should be skein invariant. However, I know of no counter-examples to the conjecture that $\sigma_L(\omega)$ is a broad oriented skein invariant for all links.

2. GOERITZ MATRICES AND THE F-POLYNOMIAL

In this section I explore the relationships between the graph of a link, its Goeritz matrix and Kauffman's polynomial invariant $F_L(a, z)$. In particular I show that the $F(a, z)$, is essentially calculable from the Goeritz matrix of a knot. This result makes use of facts about planar graphs discovered by Whitney over 50 years ago.

2.1. THE GOERITZ MATRIX AND GRAPH OF A LINK

Kauffman [10] has defined a polynomial invariant $F_L(a, z)$ of oriented links as follows:

Recall the definition of the three *Reidemeister moves,* see Figure 3.