

1.1. PRELIMINARIES

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

LINK SIGNATURE, GOERITZ MATRICES AND POLYNOMIAL INVARIANTS

by A. S. LIPSON

ABSTRACT

Lickorish and Millett introduced the notion of skein equivalence of oriented links in [15]. In the first part of this paper I show that the link signature function $\sigma_L: S^1 \rightarrow Z$ of [22], [12], etc. is a skein invariant for links with non-zero Alexander polynomial. In the second part I show that a renormalised form of Kauffman's polynomial invariant $F_L(a, z)$, well-defined on non-oriented links is calculable from the Goeritz matrix.

1. *P*-SKEINS AND SIGNATURE

In this section I present two notions of "skein equivalence" for links; "broad" skein equivalence and "narrow" skein equivalence. Narrow skein equivalence is a stronger relation (i.e. has smaller equivalence classes), but it is not clear whether it is strictly stronger. I show that the link signature function $\sigma_L: S^1 \rightarrow Z$ is a broad skein invariant for all links with non-zero Alexander polynomial. It is not known whether this result extends to links with zero Alexander polynomial, but it seems unlikely that it should.

1.1. PRELIMINARIES

I briefly recap on some standard material. See [19], [2] or [5] for further details. Let L be an oriented link. Then it is always possible to find an oriented surface F embedded in S^3 in such a way that $\partial F = L$, with the appropriate orientation. Such a surface is called a *Seifert surface* for the link L . Now let c_1, \dots, c_n be closed curves lying in S whose homology classes generate $H_1(S)$, and let c_1^+, \dots, c_n^+ be the results of pushing these curves slightly away from S in the positive direction in a collar neighbourhood of the surface. The matrix $V = (v_{ij})$, where $v_{ij} = lk(a_i, a_j^+)$, is

called a *Seifert matrix* for the link L . Of course such a matrix is not well-defined, but it is true that any two Seifert matrices for a link L are *S-equivalent*; that is, can be transformed into each other by a finite sequence of the moves

$$(1) \quad V \rightarrow PVP',$$

where P is an integral unimodular matrix, and

$$(2) \quad V \rightarrow \begin{pmatrix} V & & \\ & 0 & 1 \\ & 0 & 0 \end{pmatrix}.$$

Up to sign and multiplication by a power of t , the Alexander polynomial can be obtained by

$$(3) \quad \Delta_L(t) \doteq \det(tV - V').$$

Finally, the *determinant* of L is given by

$$(4) \quad \det(L) = i^n \det(V + V')$$

where V is an $n \times n$ square matrix, and the classical *signature* of the link L is defined by

$$(5) \quad \sigma_L = \sigma(V + V')$$

and this turns out to be a well-defined link invariant.

Tristram [22], Levine [12] and others have developed the following generalisation of the classical signature of a link: Let V be a Seifert matrix for a link L , and ω a complex number of modulus 1. Instead of considering the symmetric matrix $V + V'$, we can deal with the Hermitian matrix $H(\omega) = (1 - \bar{\omega})V + (1 - \omega)V'$. The fact that $|\omega| = 1$ enables us easily to show that the signature $\sigma(H(\omega))$ is unchanged by the moves (1) and (2) and so takes the same values for *S-equivalent* matrices. It hence provides a function $\sigma_L: S^1 \rightarrow \mathbb{Z}$ which may be regarded as a link invariant. Further, $H(\omega) = (\omega - 1)(\bar{\omega}V - V')$ so $\sigma_L(\omega)$ is continuous away from roots of the Alexander polynomial. Clearly $\sigma_L(-1)$ is twice the classical signature of a link.

By a *skein triplet* (or *oriented skein triplet*) of links I shall mean a triplet (L_+, L_-, L_0) of oriented links which are identical outside some ball $B \subset S^3$ and inside it are as shown in Figure 1.

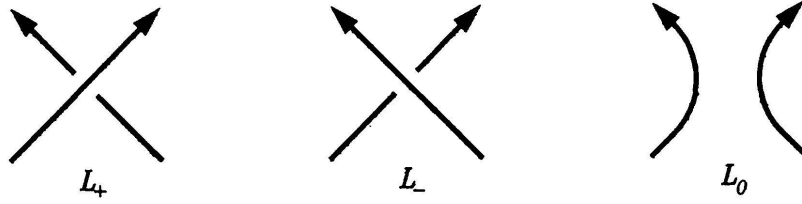


FIGURE 1

There are two equivalence relations with which we may be interested: Let R_b be the equivalence relation on the set of links generated by:

$$(6) \quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then} \quad \begin{cases} L_+ R_b L'_+, L_- R_b L'_- \Rightarrow L_0 R_b L'_0 \\ L_+ R_b L'_+, L_0 R_b L'_0 \Rightarrow L_- R_b L'_- \\ L_- R_b L'_-, L_0 R_b L'_0 \Rightarrow L_+ R_b L'_+ \end{cases}$$

I shall call this equivalence relation *broad oriented skein equivalence*. The other relation on the set of oriented links, *narrow oriented skein equivalence*, is the equivalence relation R_n generated by:

$$(7) \quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then} \quad \begin{cases} L_+ R_n L'_+, L_0 R_n L'_0 \Rightarrow L_- R_n L'_- \\ L_- R_n L'_-, L_0 R_n L'_0 \Rightarrow L_+ R_n L'_+ \end{cases}$$

It is obvious that R_b is a weaker equivalence relation than R_n (i.e. the equivalence classes are larger), but it is not clear (and I do not know) whether it is strictly weaker. By the *broad* or *narrow oriented skein of links* I refer to the set of equivalence classes of oriented links under the relation R_b or R_n (Note that in most of the literature, R_n is referred to as “skein equivalence”; R_b is not referred to at all). The polynomial invariant $P_L(l, m)$ of [15], [3] etc. may be regarded as the most general linear broad skein invariant (see [15], [16]). The fact that the value of $P_L(l, m)$ specified on the unknot U is sufficient to define its value on any link may be taken as saying that the broad oriented skein is *generated* by U . The corresponding statement for the narrow oriented skein is that specifying the values of $P_L(l, m)$ on all unlinks is sufficient to define its values on all links — the set of unlinks generates the narrow oriented skein.

1.2. SIGNATURE AND ORIENTED SKEINS

I now show that the signature function $\sigma_L(\xi)$ of any link with non-zero Alexander polynomial is a broad oriented skein invariant (It is already known that the signature $\sigma = \frac{1}{2} \sigma_L(-1)$ is a narrow skein oriented invariant