

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LINK SIGNATURE, GOERITZ MATRICES AND POLYNOMIAL INVARIANTS  
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**Kapitel:** 1.1. PRELIMINARIES  
**DOI:** <https://doi.org/10.5169/seals-57904>

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## LINK SIGNATURE, GOERITZ MATRICES AND POLYNOMIAL INVARIANTS

by A. S. LIPSON

### ABSTRACT

Lickorish and Millett introduced the notion of skein equivalence of oriented links in [15]. In the first part of this paper I show that the link signature function  $\sigma_L: S^1 \rightarrow Z$  of [22], [12], etc. is a skein invariant for links with non-zero Alexander polynomial. In the second part I show that a renormalised form of Kauffman's polynomial invariant  $F_L(a, z)$ , well-defined on non-oriented links is calculable from the Goeritz matrix.

### 1. *P*-SKEINS AND SIGNATURE

In this section I present two notions of "skein equivalence" for links; "broad" skein equivalence and "narrow" skein equivalence. Narrow skein equivalence is a stronger relation (i.e. has smaller equivalence classes), but it is not clear whether it is strictly stronger. I show that the link signature function  $\sigma_L: S^1 \rightarrow Z$  is a broad skein invariant for all links with non-zero Alexander polynomial. It is not known whether this result extends to links with zero Alexander polynomial, but it seems unlikely that it should.

#### 1.1. PRELIMINARIES

I briefly recap on some standard material. See [19], [2] or [5] for further details. Let  $L$  be an oriented link. Then it is always possible to find an oriented surface  $F$  embedded in  $S^3$  in such a way that  $\partial F = L$ , with the appropriate orientation. Such a surface is called a *Seifert surface* for the link  $L$ . Now let  $c_1, \dots, c_n$  be closed curves lying in  $S$  whose homology classes generate  $H_1(S)$ , and let  $c_1^+, \dots, c_n^+$  be the results of pushing these curves slightly away from  $S$  in the positive direction in a collar neighbourhood of the surface. The matrix  $V = (v_{ij})$ , where  $v_{ij} = lk(a_i, a_j^+)$ , is

called a *Seifert matrix* for the link  $L$ . Of course such a matrix is not well-defined, but it is true that any two Seifert matrices for a link  $L$  are *S-equivalent*; that is, can be transformed into each other by a finite sequence of the moves

$$(1) \quad V \rightarrow PVP',$$

where  $P$  is an integral unimodular matrix, and

$$(2) \quad V \rightarrow \begin{pmatrix} V & & \\ & 0 & 1 \\ & 0 & 0 \end{pmatrix}.$$

Up to sign and multiplication by a power of  $t$ , the Alexander polynomial can be obtained by

$$(3) \quad \Delta_L(t) \doteq \det(tV - V').$$

Finally, the *determinant* of  $L$  is given by

$$(4) \quad \det(L) = i^n \det(V + V')$$

where  $V$  is an  $n \times n$  square matrix, and the classical *signature* of the link  $L$  is defined by

$$(5) \quad \sigma_L = \sigma(V + V')$$

and this turns out to be a well-defined link invariant.

Tristram [22], Levine [12] and others have developed the following generalisation of the classical signature of a link: Let  $V$  be a Seifert matrix for a link  $L$ , and  $\omega$  a complex number of modulus 1. Instead of considering the symmetric matrix  $V + V'$ , we can deal with the Hermitian matrix  $H(\omega) = (1 - \bar{\omega})V + (1 - \omega)V'$ . The fact that  $|\omega| = 1$  enables us easily to show that the signature  $\sigma(H(\omega))$  is unchanged by the moves (1) and (2) and so takes the same values for *S-equivalent* matrices. It hence provides a function  $\sigma_L: S^1 \rightarrow \mathbb{Z}$  which may be regarded as a link invariant. Further,  $H(\omega) = (\omega - 1)(\bar{\omega}V - V')$  so  $\sigma_L(\omega)$  is continuous away from roots of the Alexander polynomial. Clearly  $\sigma_L(-1)$  is twice the classical signature of a link.

By a *skein triplet* (or *oriented skein triplet*) of links I shall mean a triplet  $(L_+, L_-, L_0)$  of oriented links which are identical outside some ball  $B \subset S^3$  and inside it are as shown in Figure 1.

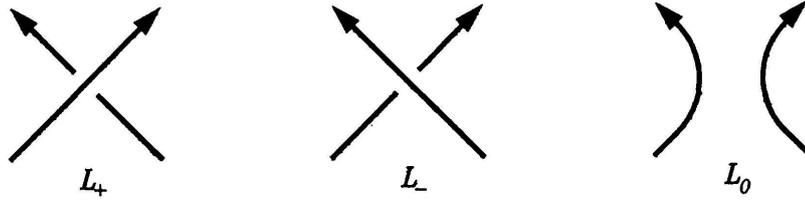


FIGURE 1

There are two equivalence relations with which we may be interested: Let  $R_b$  be the equivalence relation on the set of links generated by:

$$(6) \quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then } \begin{cases} L_+ R_b L'_+, L_- R_b L'_- \Rightarrow L_0 R_b L'_0 \\ L_+ R_b L'_+, L_0 R_b L'_0 \Rightarrow L_- R_b L'_- \\ L_- R_b L'_-, L_0 R_b L'_0 \Rightarrow L_+ R_b L'_+ \end{cases}$$

I shall call this equivalence relation *broad oriented skein equivalence*. The other relation on the set of oriented links, *narrow oriented skein equivalence*, is the equivalence relation  $R_n$  generated by:

$$(7) \quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then } \begin{cases} L_+ R_n L'_+, L_0 R_n L'_0 \Rightarrow L_- R_n L'_- \\ L_- R_n L'_-, L_0 R_n L'_0 \Rightarrow L_+ R_n L'_+ \end{cases}$$

It is obvious that  $R_b$  is a weaker equivalence relation than  $R_n$  (i.e. the equivalence classes are larger), but it is not clear (and I do not know) whether it is strictly weaker. By the *broad* or *narrow oriented skein of links* I refer to the set of equivalence classes of oriented links under the relation  $R_b$  or  $R_n$  (Note that in most of the literature,  $R_n$  is referred to as “skein equivalence”;  $R_b$  is not referred to at all). The polynomial invariant  $P_L(l, m)$  of [15], [3] etc. may be regarded as the most general linear broad skein invariant (see [15], [16]). The fact that the value of  $P_L(l, m)$  specified on the unknot  $U$  is sufficient to define its value on any link may be taken as saying that the broad oriented skein is *generated* by  $U$ . The corresponding statement for the narrow oriented skein is that specifying the values of  $P_L(l, m)$  on all unlinks is sufficient to define its values on all links — the set of unlinks generates the narrow oriented skein.

## 1.2. SIGNATURE AND ORIENTED SKEINS

I now show that the signature function  $\sigma_L(\xi)$  of any link with non-zero Alexander polynomial is a broad oriented skein invariant (It is already known that the signature  $\sigma = \frac{1}{2} \sigma_L(-1)$  is a narrow skein oriented invariant