Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE POMPEIU PROBLEM REVISITED

Autor: Bagchi, S. C. / Sitaram, A.

Kapitel: 7. Pompeiu property for two-sided translations on groups

DOI: https://doi.org/10.5169/seals-57903

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 02.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

7. Pompeiu property for two-sided translations on groups

In this section we consider the Pompeiu problem in the setting where X is one of the groups M(2) or $SL(2, \mathbf{R})$ and the group action is that of two-sided translations. We start with a general definition.

Definition 7.1. Let G be a locally compact unimodular group. A measurable relatively compact subset $E \subseteq G$ of positive Haar measure is said to have the (two-sided) Pompeiu property if for $f \in C(G)$,

$$\int_{g_1 E g_2} f(x) dx = 0 \quad \forall g_1, g_2 \in G$$

implies $f \equiv 0$ where the integration is with respect to the Haar measure.

Except for the two groups mentioned the problem in general appears to be untractable at the moment, the primary reason being that there has not been much progress with the related problem of two-sided spectral analysis. If, however, we consider only functions $f \in L^1(G) \cap C(G)$ the problem considered above becomes easier and some investigations have been made in this restricted set-up (see [20], [21], [24]). Our treatment of M(2) and $SL(2, \mathbb{R})$ relies on the work of Weit ([32]) and Ehrenpreis and Mautner ([13], [14]) on spectral analysis and synthesis on M(2) and $SL(2, \mathbb{R})$ respectively. (See also [4], [31] in this connection.)

§ 7.1. We introduce a class of representations of the group M(2). As in Section 3, we write $M(2) = \{(x, \sigma) : x \in \mathbb{R}^2, \sigma \in SO(2, \mathbb{R})\}$ where $(x, \sigma) \cdot (x', \sigma')$ = $(x + \sigma x', \sigma \sigma')$ is the group multiplication and an element (x, σ) acts on $y \in \mathbb{R}^2$ by the rule $(x, \sigma)y = \sigma y + x$. Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and we define representations Π_{λ} on the space $L^2(S^1)$ where $S^1 \subseteq \mathbb{R}^2$ is the unit circle: for $(x, A) \in M(2)$,

$$\Pi_{\lambda}(x, A)f(w) = e^{i\lambda x \cdot w} f(A^{-1}w) \quad f \in L^{2}(S), \quad w \in S^{1},$$

where \cdot is the inner product in \mathbb{R}^2 . The Π_{λ} 's are related to the representations of M(2) on the eigenspaces of the Laplacian Δ on \mathbb{R}^2 and are irreducible (see [16], p. 12). If λ is real, then Π_{λ} is seen to be a unitary representation. The only other irreducible unitary representation (up to equivalence) of M(2) are the characters:

$$\chi_n(x, A) = e^{in\theta}, \quad x \in \mathbf{R}^2, A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

We now come to our result for the group M(2).

Theorem 7.1. A measurable relatively compact subset $E \subseteq M(2)$ of positive Haar measure has the Pompeiu property if and only if

- (i) the operators $\int_{E} \Pi_{\lambda}(x) dx \neq 0$ for each $\lambda \in \mathbb{C}$, $\lambda \neq 0$;
- (ii) $\int_{E} \chi_{n}(x) dx \neq 0$ for each integer n;

where the integrals are with respect to the Haar measure on M(2).

Proof. If Π is a continuous irreducible representation of G on a Hilbert space and $\Pi(1_E)=0$, then for a suitably chosen matrix element $f:x\to <\Pi(x)v_1$, $v_2>$, we have $\int_{g_1Eg_2}f(x)dx=0$ for all g_1 , g_2 . The only if part now follows. To prove the if part we consider the two-sided ideal of C(M(2)):

$$\mathscr{U} = \{ f \in C(M(2)) : \int_{g_1 E g_2} f dx = 0 \text{ for all } g_1, g_2 \in M(2) \}.$$

Assume $\mathscr{U} \neq \{0\}$. We shall prove that either for some n, $\int_E \chi_n(x) dx = 0$, or for some $\lambda \neq 0$, $\int_E \Pi_{\lambda}(x) dx = 0$. Since $\mathscr{U} \neq \{0\}$, by Weit's theorem ([32], Theorem 1), either $\chi_n \in \mathscr{U}$ for some n or there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$ such that the functions $g_{\lambda}(x, A) = e^{i(\lambda_1 x_1 + \lambda_2 x_2)}$, where $(x, A) \in M(2)$ $(x = (x_1, x_2))$, belongs to \mathscr{U} . In case $\chi_n \in \mathscr{U}$ we immediately get the desired result. So let for λ as above, $g_{\lambda} \in \mathscr{U}$. As noted by Weit if $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$ and $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$, then $g_{\mu} \in \mathscr{U}$. We choose $\mu_1 = \sqrt{\lambda_1^2 + \lambda_2^2} \cos \theta$, $\mu_2 = \sqrt{\lambda_1^2 + \lambda_2^2} \sin \theta$ for some $\theta \in \mathbb{R}$, so that

$$g_{u}(x, A) = \exp(i\sqrt{\lambda_{1}^{2} + \lambda_{2}^{2}} x \cdot w), \quad (x, A) \in M(2),$$

where $w = (\cos \theta, \sin \theta)$. Since $g_{\mu} \in \mathcal{U}$ for all $\theta \in \mathbf{R}$ and since \mathcal{U} is closed, we have $h \in \mathcal{U}$ where

$$h(x, A) = \int_{S^1} \exp(i\sqrt{\lambda_1^2 + \lambda_2^2} x \cdot w) dw, \quad (x, A) \in M(2).$$

But $h(x, A) = \langle \Pi_z(x, A)1, 1 \rangle, (x, A) \in M(2)$ where $z = \sqrt{\lambda_1^2 + \lambda_2^2}$ and so we have

$$\int_{g_1 E g_2} < \Pi_z(g) 1, \ 1 > dg = 0$$

for all g_1 , $g_2 \in G$. This means

$$\int_{E} \langle \Pi_{z}(g_{1})\Pi_{z}(g)\Pi_{z}(g_{2})1, 1 \rangle dg = 0, \quad g_{1}, g_{2} \in G.$$

By taking adjoints and using the fact $\Pi_z(g)^* = \Pi_{-\overline{z}}(g^{-1})$,

$$\int_{E} \langle \Pi_{z}(g)\Pi_{z}(g_{2})1, \Pi_{-\overline{z}}(g_{1}^{-1})1 \rangle dg = 0, \quad g_{1}, g_{2} \in G.$$

Since Π_z and $\Pi_{-\overline{z}}$ are irreducible, 1 is a cyclic vector in $L^2(S^1)$ for both these representation and it will, therefore, follow

$$\int_{E} <\Pi_{z}(g)u, v> dg = 0$$

for all $u, v \in L^2(S^1)$. This shows that $\Pi_z(1_E) = 0$ and the theorem is proved.

§ 7.2. For simplicity, we consider the group

$$PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{+1, -1\}$$

rather than $SL(2, \mathbf{R})$ itself. For each $\lambda \in \mathbf{C}$, $SL(2, \mathbf{R})$ has the so-called principal series representation Π_{λ} (which are continuous representations realised on a Hilbert space). It is well known, that but for a countable set of λ 's Π_{λ} is irreducible. Thus the set of representations $\{\Pi_{\lambda} : \lambda \in \mathbf{C}, \Pi_{\lambda} \text{ irreducible}\}$ along with another countable family of representations called the discrete series and all irreducible finite-dimensional representations account for all so-called "topologically completely irreducible" Banach representations (upto equivalence) of $PSL(2, \mathbf{R})$. We now state our theorem for $PSL(2, \mathbf{R})$:

THEOREM 7.2. A relatively compact measurable subset $E \subseteq PSL(2, \mathbf{R})$ of positive Haar measure has the Pompeiu property if and only if for each topologically completely irreducible Banach representation Π of $PSL(2, \mathbf{R})$, $\Pi(1_E) \neq 0$.

Sketch of Proof. As with Theorem 7.1, the proof relies on spectral analysis of two-sided ideals of $C^{\infty}(G)$ due to Ehrenpreis and Mautner ([13], [14]): If $0 \neq \mathcal{U} \subseteq C^{\infty}(G)$ is a closed two-sided ideal in $C^{\infty}(G)$, then there exists $\lambda \in \mathbb{C}$ and a nontrivial function of the form $\phi_{n,m}^{\lambda} \colon x \to \langle \Pi_{\lambda}(x)e_n, e_m \rangle$ is in \mathcal{U} . Here $\{e_j\}$ is an orthonormal basis of \mathcal{H}_{λ} , the representation space for Π_{λ} and moreover, $\Pi_{\lambda}(k)e_j = \chi_j(k)e_j$ where $\{\chi_j\}$ are the characters of the maximal compact subgroup $SO(2, \mathbb{R})/\{+1, -1\}$ of $PSL(2, \mathbb{R})$ and $k \in SO(2, \mathbb{R})/\{+1, -1\}$. To prove the theorem, we define \mathcal{U} as in Theorem 7.1. If $\mathcal{U} \neq \{0\}$, then appealing to the result quoted above we get $\phi_{n,m}^{\lambda} \in \mathcal{U}$ for some $\lambda \in \mathbb{C}$ and n, m integers. If the corresponding Π_{λ} is irreducible,

then the argument is as in Theorem 7.1 and we can prove $\Pi_{\lambda}(1_E) = 0$. If Π_{λ} is not irreducible, then depending on n, m we can find a discrete series representation or an irreducible finite dimensional representation Π occurring either as a subrepresentation or as a subquotient of Π_{λ} for which $\Pi(1_E) = 0$. To do this, we need the exact G-module structure of Π_{λ} which in the case of $PSL(2, \mathbf{R})$ is available (see for example [14]).

§ 7.3. Consider the case $G = \mathbb{R}^2$ and G acting on itself by translations. In this case, Brown, Schreiber and Taylor have proved that there are no Pompeiu sets ([12]). In view of this it would be natural to ask if there are sets E satisfying the conditions of Theorem 7.1 at all. Identify \mathbb{R}^2 with G/K where G = M(2) and $K = SO(2, \mathbb{R})$; if $E \subseteq G/K$ then one can show that the condition of Brown, Schreiber and Taylor considered in Section 3 is equivalent to the condition $\Pi_{\lambda}(1_{\tilde{E}}) \neq 0$ for $\lambda \in \mathbb{C}$, $\lambda \neq 0$. (A special case of this observation is also made in [30]). Hence by the discussion in Section 3, there are plenty of sets E with this property. As we have seen, topologically $G \approx \mathbb{R}^2 \times SO(2, \mathbb{R})$. We now observe that if E is chosen as above in \mathbb{R}^2 and E is a suitably chosen arc in E for all E is chosen as above in E and E is a subset of E satisfies E satisfies E for all E is chosen as well as E and E for all E is chosen as a power of E and E for all E is chosen as a power of E and E is a suitably chosen arc in E for all E is chosen as well as E and E for all E is irrational modulo E and E is irrational modulo E and E is irrational modulo E and E is irrational modulo E is irrational modulo E and E is irrational modulo E is irrational modulo E in the set E is chosen as a subset of E is irrational modulo E is irrational modulo E in the set E is chosen as a subset of E is irrational modulo E in the set E is chosen as a subset of E is irrational modulo E in the set E is chosen as a subset of E is irrational modulo E in the set E is an interval in the set E in the set E is chosen as a subset of E is an interval in the set E in the set E is an interval in the set E in the set E in the set E is an interval in the set E in the set E in the set E is an interval in the set E is an interva

8. Concluding remarks

In this paper, we have restricted our attention to the Pompeiu property for a single set E. One can also consider the Pompeiu property for a collection of sets or distributions of compact support as in [9], [12]. There are also closely related properties such as the Morera property — see [12] for details.

As pointed out earlier the Pompeiu problem becomes easier if one considers only integrable functions. Investigations under this assumption have been done, for example, in [2], [20], [24] and [28]. If one only considers integrable functions one need not restrict oneself to relatively compact sets. Moreover, considering integrable functions is equivalent to considering finite complex measures. Thus for G a locally compact abelian group a Borel subset $E \subseteq G$ is said to be a determining set for finite complex measures if for a finite complex measure μ on G, $\mu(gE) = 0$ for all $g \in G$ implies $\mu = 0$.

For locally compact abelian groups it is easy to see that a set of finite Haar measure is a determining set for finite complex measures if and only if