

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 36 (1990)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE POMPEIU PROBLEM REVISITED
Autor: Bagchi, S. C. / Sitaram, A.
Kapitel: 3. POMPEIU PROBLEM FOR THE M(2) ACTION ON \mathbb{R}^2
DOI: <https://doi.org/10.5169/seals-57903>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 23.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$\begin{aligned}
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}(f))^\vee(0) \\
&\quad (\text{where } g^\vee(x) = g(-x), g \in \mathcal{E}(\mathbf{R}^2), x \in \mathbf{R}^2), \\
&= \Sigma(S) * ((\tilde{\Sigma}f)^\vee * \Sigma(T)^\vee)(0) \\
&= \Sigma(S) * (\tilde{\Sigma}(f) * \Sigma(T)) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even} \\
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}f)(0) \\
&= \langle \Sigma(S * T), \tilde{\Sigma}f \rangle \\
&\quad (\text{using } \Sigma S * \Sigma T = \Sigma S * T) \\
&= \langle S * T, f \rangle \\
&= S * T * f(0) \text{ as } f \text{ is even} \\
&= \langle S, T * f \rangle.
\end{aligned}$$

On the other hand,

$$\langle \Sigma(S), \tilde{\Sigma}(T * f) \rangle = \langle S, T * f \rangle.$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

THEOREM 2.4. *Let \mathcal{V} be a closed nonzero subspace of $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$ such that for all $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ and $f \in \mathcal{V}$, $T * f \in \mathcal{V}$. Then there exists $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathcal{V}$.*

Proof. Consider the closed and nontrivial subspace M of $\mathcal{E}(\mathbf{R})_e$ such that $\tilde{\Sigma}(\mathcal{V}) = M$. By Lemma 2.3, M is closed under convolution with elements $S \in \mathcal{E}'(\mathbf{R})_e$. By the remarks following Theorem 2.1 now, there exists $\lambda \in \mathbf{C}$ such that the functions $\Psi_\lambda \in M$, where $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$, $x \in \mathbf{R}$. A simple calculation now shows

$$\langle \phi_\lambda, f \rangle = \langle \Psi_\lambda, \Sigma f \rangle \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}} \subseteq \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}.$$

Thus $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$ and hence $\phi_\lambda \in \mathcal{V}$.

3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON \mathbf{R}^2

The Euclidean motion group $M(2)$ is the semidirect product of \mathbf{R}^2 with the rotation group $SO(2, \mathbf{R})$.

$$M(2) = \{(x, \sigma) : x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$$

where

$$(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma \sigma')$$

is the group multiplication and an element (x, σ) acts on $y \in \mathbf{R}^2$ by the rule $(x, \sigma)y = \sigma y + x$.

Let E be a relatively compact subset of \mathbf{R}^2 of positive Lebesgue measure. If $f \in C(\mathbf{R}^2)$, the space of continuous functions on \mathbf{R}^2 , the vanishing of the integrals

$$\int_{gE} f(x)dx = 0, \text{ for all } g \in M(2)$$

i.e. $\int_{\sigma E + y} f(x)dx = 0, \text{ for all } \sigma \in SO(2, \mathbf{R}), y \in \mathbf{R}^2$

can be restated as $f * \check{1}_{\sigma E} \equiv 0$, for all $\sigma \in SO(2, \mathbf{R})$ or, equivalently $f^\sigma * \check{1}_E \equiv 0$ for all $\sigma \in SO(2, \mathbf{R})$, where $f^\sigma(x) = f(\sigma x)$ and $\check{1}_E(x) = 1_E(-x)$, $x \in \mathbf{R}^2$. We write

$$\mathcal{U} = \{f \in \mathcal{E}(\mathbf{R}^2) : f^\sigma * \check{1}_E = 0 \text{ for all } \sigma \in SO(2, \mathbf{R})\}.$$

From elementary smoothing arguments, it follows that E has the Pompeiu property if and only if $\mathcal{U} = \{0\}$. \mathcal{U} is a closed subspace of $\mathcal{E}(\mathbf{R}^2)$ which is invariant under translation and rotation. Let again

$$\mathcal{V} = \{f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}} : f * \check{1}_E = 0\}.$$

Then $\mathcal{V} \subseteq \mathcal{U}$, \mathcal{V} is a closed subspace of $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$ and $T * \mathcal{V} \subseteq \mathcal{V}$ for all $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$.

We now prove the main theorem of [12] mentioned in the Introduction. (However, we restrict ourselves to indicator functions of sets, rather than general distributions of compact support.)

THEOREM 3.1 (Brown, Schreiber and Taylor). *A relatively compact subset $E \subseteq \mathbf{R}^2$ of positive Lebesgue measure does not have the Pompeiu property if and only if there exists $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that*

$$\hat{1}_E(z_1, z_2) = 0 \text{ whenever } z_1^2 + z_2^2 = \alpha^2,$$

where $\hat{1}_E$ is the Laplace-Fourier transform of the characteristic function 1_E of E .

Proof. The *if* part is immediate; for instance, take any $z = (z_1, z_2)$ such that $z_1^2 + z_2^2 = \alpha^2$ and consider the function $e^{iz \cdot x}$. To prove the *only if* part, suppose E has the Pompeiu property. Let \mathcal{U} and \mathcal{V} be defined as above; by assumption we have $\mathcal{U} \neq \{0\}$. We shall now prove that $\mathcal{V} \neq \{0\}$. Choose $f \in \mathcal{U}$ with $f(0) \neq 0$ (this is possible as \mathcal{U} is translation-invariant). Define

$$h(y) = \int_{SO(2, \mathbf{R})} f(\sigma y) d\sigma, \quad y \in \mathbf{R}^2.$$

As \mathcal{U} is $SO(2, \mathbf{R})$ -invariant, the function $h \in \mathcal{U}$. Further, h is a radial function by definition, so $h \in \mathcal{V}$. But then $h(0) = f(0) \neq 0$. Thus $\mathcal{V} \neq \{0\}$ and by Theorem 2.2, we have $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathcal{V}$. Further, ϕ_0 is the constant function and hence ϕ_0 cannot belong to $\mathcal{V} \subseteq \mathcal{U}$. So $\lambda \neq 0$ and $\phi_\lambda \in \mathcal{V}$ and, in particular, $\phi_\lambda * 1_E(0) = 0$. In the notation of Section 2, this means $\mathcal{G}1_E(\lambda) = 0$ and hence $\hat{1}_E(\lambda, 0) = 0$. The $SO(2, \mathbf{R})$ -invariance of \mathcal{U} now shows that $\hat{1}_E$ vanishes on $SO(2, \mathbf{R}) \cdot (\lambda, 0)$. The analyticity argument in Lemma 2.2 will now prove that $\hat{1}_E$ vanishes at all (z_1, z_2) where $z_1^2 + z_2^2 = \lambda^2$. This proves the theorem.

The condition in Theorem 3.1 can also be given a representation theoretic interpretation in terms of the so-called class-1 principal series representation of $M(2)$ — see Section 7.3 for a more precise statement. As we remarked earlier, the condition of the theorem is verifiable only for sets having strong geometric properties. We quote two results from [12] without proof.

THEOREM 3.2 (Brown, Schreiber and Taylor). *The ellipse*

$$E = \left\{ (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

has the Pompeiu property if and only if $a, b > 0$ and $a \neq b$.

When $a = b > 0$, D is the disc and we have

$$\hat{1}_D(z_1, z_2) = \text{const. } J_1(\sqrt{z_1^2 + z_2^2}) / \sqrt{z_1^2 + z_2^2}, \quad (z_1, z_2) \in \mathbf{C}^2,$$

where J_1 is the Bessel function. Since there are infinitely many zeros of J_1 , D does not have the Pompeiu property. The next theorem, obtained through a careful estimate ([12]) needs a definition.

Definition 3.3. Let $\Gamma = \Gamma(t)$, $-1 \leq t \leq 1$ be a Lipschitz curve in \mathbf{R}^2 with well defined (a.e.) unit tangent vectors $T(t) = \Gamma'(t) / |\Gamma'(t)|$. The point $p = \Gamma(0)$ is a corner of Γ if both the right and the left limits of $T(t)$ as $t \rightarrow 0$ exist and are not multiples of each other.

THEOREM 3.4 (Brown, Schreiber and Taylor). *Let Ω be a compact connected subset of \mathbf{R}^2 . Suppose that there is a half-plane H and a unique point $p \in \Omega \cap H$ of maximal distance from the boundary ∂H of H . If the boundary of Ω near p is given by a Lipschitz curve with a corner at p then Ω has the Pompeiu property.*

Let now Ω be a bounded Borel subset of the plane of positive measure and suppose that Ω does not have the Pompeiu property. By Theorem 3.1, $\hat{1}_\Omega$ vanishes on the algebraic variety $M_\alpha = \{(z_1, z_2) : z_1^2 + z_2^2 = \alpha^2\}$ for some $\alpha \neq 0$. As observed in [33] and [34], $\hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$ is now an entire function and standard Paley-Wiener theorem yields the following proposition.

PROPOSITION 3.5. *If Ω is a bounded Borel subset of \mathbf{R}^2 of positive measure with $\hat{1}_\Omega$ vanishing on $M_\alpha, \alpha \neq 0$, then the function $g(z_1, z_2) = \hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$ is an entire function on \mathbf{C}^2 which is the Laplace-Fourier transform of a distribution of compact support.*

Proposition 3.5 immediately gives rise to a partial differential equation. For, if T is the distribution whose Fourier transform is g , then from

$$(z_1^2 + z_2^2 - \alpha^2)g(z_1, z_2) = \hat{1}_\Omega(z_1, z_2)$$

we have

$$(3.1) \quad \Delta T + \alpha^2 T = -1_\Omega$$

where Δ is the Laplacian. Conversely, if there exists a distribution T of compact support satisfying the equation (3.1), then $\hat{1}_\Omega$ vanishes on M_α and hence Ω does not have the Pompeiu property. We also remark that if Ω is, further, a bounded simply connected open set and the equation (3.1) has a solution, then α^2 is necessarily a positive real number as can be seen from a simple Green's theorem argument (see [34] for a proof). The equation has been studied in [3], [33] and [34]. In [33] it was proved that a solution of (3.1), if it exists is actually a function. We shall discuss some more of these results in the next section. We end the present section by quoting the main theorem of [34]. This result extends Theorem 3.4 and, barring sets of rotational symmetry all known sets failing to have the Pompeiu property are covered by this result. For a bounded subset $\Omega \subseteq \mathbf{R}^2$, we denote by $\partial^* \Omega$ the boundary of the unbounded component.

THEOREM 3.5 (S. A. Williams). *Let Ω be a bounded open subset such that the equation $\Delta T + \alpha^2 T = -1_\Omega$ has a function solution of compact support for some $\alpha > 0$. Let R, K, L be positive real numbers such that $L > KR$. Assume that for $P \in \partial^* \Omega$ there exists a coordinate system (x, y) around P so that*

- (i) $Q = (-R, R) \times (-L, L)$ intersects $\partial\Omega$ in the graph $y = f(x)$ of a Lipschitz function f with Lipschitz constant K , and
- (ii) $Q \cap \Omega = \{(x, y) : |x| < R \text{ and } f(x) < y < L\}$.

Then f is real analytic in a neighbourhood of P .

Thus if we restrict ourselves to the class \mathcal{D} of simply connected bounded open sets with Lipschitz boundary then $\Omega \in \mathcal{D}$ can fail to have the Pompeiu property only if $\partial\Omega$ is real analytic.

4. A LONG-STANDING CONJECTURE !

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

Conjecture. If $\Omega \subseteq \mathbf{R}^2$ is in the class \mathcal{D} described above and if Ω does not have the Pompeiu property, then Ω is a disc.

As pointed out before, the work of Williams shows that is is enough to consider Ω with $\partial\Omega$ real analytic. For $\Omega \in \mathcal{D}$, the existence of (a necessarily positive) α^2 for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive γ for which the following overdetermined system has a solution.

$$(4.1) \quad \Delta T + \gamma T = 0 \quad \text{on } \Omega$$

$$T = \text{constant} \neq 0 \quad \text{on } \partial\Omega, \partial T / \partial n \equiv 0 \quad \text{on } \partial\Omega$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for $\Omega \in \mathcal{D}$, there exists $\gamma > 0$ for which (4.1) admits a solution, then Ω is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If Ω is a bounded connected open set with smooth boundary on which

$$\begin{aligned} \Delta u &= -1 \quad \text{on } \Omega \\ u &= 0, \partial u / \partial n = \text{constant} \quad \text{on } \partial\Omega \end{aligned}$$

has a function solution, then Ω must be a disc.