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provide a brief survey of the literature on some allied problems. Though extensive, our bibliography is far from complete. We refer the reader to the bibliographies in [9], [12] and [35].

2. SPECTRAL ANALYSIS OF RADIAL FUNCTIONS

We denote by $\mathcal{E}(\mathbf{R}^n)$, the space of C^∞ functions on \mathbf{R}^n with the usual topology and by $\mathcal{E}'(\mathbf{R}^n)$, the dual space of distributions of compact support with the strong topology — both Fréchet-Montel and hence reflexive spaces. $C_c^\infty(\mathbf{R}^n)$ is the space of C^∞ -functions of compact support. For a space of functions or distributions \mathcal{F} , we denote the usual action of an element σ of the orthogonal group $O(n, \mathbf{R})$ by the notation $f \rightarrow f^\sigma$. \mathcal{F}_{rad} will stand for the space of those $f \in \mathcal{F}$ which are invariant under $O(n, \mathbf{R})$, i.e., $f^\sigma = f$ for all $\sigma \in O(n, \mathbf{R})$. $\mathcal{E}'(\mathbf{R}^n)_{\text{rad}}$ is a closed subspace of $\mathcal{E}'(\mathbf{R}^n)$ and the spaces $\mathcal{E}(\mathbf{R}^n)_{\text{rad}}$ and $\mathcal{E}'(\mathbf{R}^n)_{\text{rad}}$ are (strong) duals of each other. In the case $n = 1$, even functions are the analogues of radial functions and we write \mathcal{F}_e to mean \mathcal{F}_{rad} . Though our considerations in this section hold for all $n \geq 2$, we shall restrict ourselves to the case $n = 2$ to keep the exposition simple.

We start with a slightly weaker version of the classical theorem of L. Schwartz ([23]).

THEOREM 2.1 (L. Schwartz's theorem on spectral analysis). *Let \mathcal{U} be a nontrivial closed subspace of $\mathcal{E}(\mathbf{R})$, which is closed under translations, then \mathcal{U} contains an exponential function $e^{i\lambda x}$ for some $\lambda \in \mathbf{C}$.*

As pointed out in [1] an immediate corollary of the theorem is: If \mathcal{U} is a nontrivial closed subspace of $\mathcal{E}(\mathbf{R})_e$ which is closed under convolution against all $T \in \mathcal{E}'(\mathbf{R})_e$, then \mathcal{U} contains a function of the form $\psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$.

We now introduce a family of functions on \mathbf{R}^2 which is central to spectral analysis of radial functions. For $\lambda \in \mathbf{C}$, define

$$\phi_\lambda(x) = \int_{|w|=1} e^{-i\lambda(x \cdot w)} dw, \quad x \in \mathbf{R}^2$$

where the integral is with respect to the normalised Lebesgue measure on the unit circle. Here $x \cdot w$ is the usual inner product. It is immediate that ϕ_λ is a radial function for each $\lambda \in \mathbf{C}$. For $f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}$, we define a transform (sometimes called the Bessel transform):

$$\mathcal{G}f(\lambda) = \int_{\mathbf{R}^2} \phi_\lambda(x) f(x) dx, \quad \lambda \in \mathbf{C}.$$

We see that if \hat{f} is the Fourier-Laplace transform of $f \in C_c^\infty(\mathbf{R}^2)$, i.e.,

$$\hat{f}(z_1, z_2) = \int_{\mathbf{R}^2} e^{-i(z \cdot x)} f(x) dx, \quad z = (z_1, z_2) \in \mathbf{C}^2,$$

then we have

$$(2.1) \quad \mathcal{G}f(\lambda) = \hat{f}(\lambda, 0), \quad \lambda \in \mathbf{C}, \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}.$$

Both the transforms \mathcal{G} defined above and the Fourier-Laplace transform have their obvious extension to $\mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$. We have for $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$,

$$\mathcal{G}T(\lambda) = T(\phi_\lambda), \quad \lambda \in \mathbf{C}$$

$$\hat{T}(z_1, z_2) = T(e_z), \quad z \in \mathbf{C}^2$$

where $e_z(x) = e^{-i(z_1 x_1 + z_2 x_2)} = e^{-i(z \cdot x)}$. We again have

$$\mathcal{G}T(\lambda) = \hat{T}(\lambda, 0), \quad \lambda \in \mathbf{C}.$$

By applying the Paley-Wiener theorem we are able to obtain a description of the function space $\mathcal{X} = \{\mathcal{G}T : T \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}\}$.

LEMMA 2.2. \mathcal{X} is the space of even entire functions f on \mathbf{C} such that for some constants, c, N and A (depending on f),

$$|f(\lambda)| \leq C(1 + |\lambda|)^N e^{A|\text{Im } \lambda|}, \quad \lambda \in \mathbf{C}.$$

Proof. By the Paley-Wiener theorem an entire function $\phi = \hat{T}$ for some $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ if and only if for some $C, N, A > 0$,

$$|\phi(z)| \leq C(1 + |z|)^N e^{A|\text{Im } z|}$$

and

$$\phi(z) = \phi(\sigma z)$$

for all $z = (z_1, z_2) \in \mathbf{C}^2$ and $\sigma \in SO(2, \mathbf{R})$ (here $\text{Im } z = (\text{Im } z_1, \text{Im } z_2)$). The latter condition is equivalent to saying $\phi(z) = \phi(z')$ whenever $z_1^2 + z_2^2 = z_1'^2 + z_2'^2$. To see this, consider, for each $\alpha \in \mathbf{C}$,

$$M_\alpha = \{z : z_1^2 + z_2^2 = \alpha^2\}.$$

If $\alpha \neq 0$, M_α is a connected analytic submanifold of \mathbf{C}^2 of complex dimension 1 and $SO(2, \mathbf{R})(\alpha, 0, \dots, 0)$ is a real submanifold of M_α of dimension 1 on which the analytic function ϕ is given to be constant. This forces ϕ to be a constant on M_α . A modification of the argument is necessary for $\alpha = 0$.

The lemma now follows from the simple observation that if $\lambda^2 = z_1^2 + z_2^2$, then $(\operatorname{Im} \lambda)^2 \leq (\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2$ and from the relation 2.1.

A straight-forward application of the one-dimensional Paley-Wiener theorem for even distributions of compact support will show that \mathcal{X} is also equal to

$$\{\hat{T}: T \in \mathcal{E}'(\mathbf{R})_e\}.$$

This identification allows us to define the linear map Σ by

$$\begin{aligned}\Sigma: \mathcal{E}'(\mathbf{R}^2)_{\text{rad}} &\rightarrow \mathcal{E}'(\mathbf{R}^2)_e \\ (\Sigma T)^\wedge(\lambda) &= \mathcal{G}T(\lambda), \quad \lambda \in \mathbf{C}.\end{aligned}$$

Σ is one-to-one and onto. Moreover, we have the following description of the strong topology in $\mathcal{E}'(\mathbf{R}^n)$ (see [12], prop. 2.1): $T_n \rightarrow T$ if and only if

- (i) $\hat{T}_n \rightarrow \hat{T}$ uniformly on compact sets along with the derivatives and
- (ii) $\hat{T}_n, n \geq 1$ satisfy the uniform Paley-Wiener condition:

$$|\hat{T}_n(z)| \leq C(1 + |z|)^N e^{A|\operatorname{Im} z|}, \quad z \in \mathbf{C}^n.$$

for some $C, N, A > 0$. This description coupled with the observation in the last step of the proof of Lemma 2.2 gives that Σ is a topological linear isomorphism between $\mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ and $\mathcal{E}'(\mathbf{R})_e$ preserving convolution.

On using the reflexivity of $\mathcal{E}(\mathbf{R})_e$ and $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$ we now get the map $\tilde{\Sigma}$:

$$\begin{aligned}\tilde{\Sigma}: \mathcal{E}(\mathbf{R}^2)_{\text{rad}} &\rightarrow \mathcal{E}(\mathbf{R})_e, \\ \langle \Sigma(T), \tilde{\Sigma}(f) \rangle &= \langle T, f \rangle \quad T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}, f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}}\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the pairing of dual spaces.

We now have the following useful lemma:

LEMMA 2.3. *With the notation above, we have*

$$\Sigma(T) * \tilde{\Sigma}(f) = \tilde{\Sigma}(T * f)$$

for all $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ and $f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}}$, where $*$ denotes the usual convolution on \mathbf{R}^2 .

Proof. Let $S \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$. Consider

$$\langle \Sigma(S), (\Sigma(T) * \tilde{\Sigma}(f)) \rangle$$

$$\begin{aligned}
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}(f))^{\vee}(0) \\
&\quad (\text{where } g^{\vee}(x) = g(-x), g \in \mathcal{E}(\mathbf{R}^2), x \in \mathbf{R}^2), \\
&= \Sigma(S) * ((\tilde{\Sigma}f)^{\vee} * \Sigma(T)^{\vee})(0) \\
&= \Sigma(S) * (\tilde{\Sigma}(f) * \Sigma(T)) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even} \\
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}f)(0) \\
&= \langle \Sigma(S * T), \tilde{\Sigma}f \rangle \\
&\quad (\text{using } \Sigma S * \Sigma T = \Sigma S * T) \\
&= \langle S * T, f \rangle \\
&= S * T * f(0) \text{ as } f \text{ is even} \\
&= \langle S, T * f \rangle.
\end{aligned}$$

On the other hand,

$$\langle \Sigma(S), \tilde{\Sigma}(T * f) \rangle = \langle S, T * f \rangle.$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

THEOREM 2.4. *Let \mathcal{V} be a closed nonzero subspace of $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$ such that for all $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ and $f \in \mathcal{V}$, $T * f \in \mathcal{V}$. Then there exists $\lambda \in \mathbf{C}$ such that $\phi_{\lambda} \in \mathcal{V}$.*

Proof. Consider the closed and nontrivial subspace M of $\mathcal{E}(\mathbf{R})_e$ such that $\tilde{\Sigma}(\mathcal{V}) = M$. By Lemma 2.3, M is closed under convolution with elements $S \in \mathcal{E}'(\mathbf{R})_e$. By the remarks following Theorem 2.1 now, there exists $\lambda \in \mathbf{C}$ such that the functions $\Psi_{\lambda} \in M$, where $\Psi_{\lambda}(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$, $x \in \mathbf{R}$. A simple calculation now shows

$$\langle \phi_{\lambda}, f \rangle = \langle \Psi_{\lambda}, \Sigma f \rangle \quad f \in C_c^{\infty}(\mathbf{R}^2)_{\text{rad}} \subseteq \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}.$$

Thus $\tilde{\Sigma}\phi_{\lambda} = \Psi_{\lambda}$ and hence $\phi_{\lambda} \in \mathcal{V}$.

3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON \mathbf{R}^2

The Euclidean motion group $M(2)$ is the semidirect product of \mathbf{R}^2 with the rotation group $SO(2, \mathbf{R})$.