provide a brief survey of the literature on some allied problems. Though extensive, our bibliography is far from complete. We refer the reader to the bibliographies in [9], [12] and [35].

2. Spectral analysis of radial functions

We denote by $\mathscr{E}(\mathbf{R}^n)$, the space of C^∞ functions on $\mathbf{R}^n$ with the usual topology and by $\mathscr{E}'(\mathbf{R}^n)$, the dual space of distributions of compact support with the strong topology — both Fréchet-Montel and hence reflexive spaces. $C_c^\infty(\mathbf{R}^n)$ is the space of C^∞-functions of compact support. For a space of functions or distributions $\mathscr{F}$, we denote the usual action of an element σ of the orthogonal group $O(n, \mathbf{R})$ by the notation $f \to f^\sigma$. $\mathscr{F}_{\text{rad}}$ will stand for the space of those $f \in \mathscr{F}$ which are invariant under $O(n, \mathbf{R})$, i.e., $f^\sigma = f$ for all $\sigma \in O(n, \mathbf{R})$. $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ is a closed subspace of $\mathscr{E}'(\mathbf{R}^n)$ and the spaces $\mathscr{E}(\mathbf{R}^n)_{\text{rad}}$ and $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ are (strong) duals of each other. In the case $n = 1$, even functions are the analogues of radial functions and we write $\mathscr{F}_e$ to mean $\mathscr{F}_{\text{rad}}$. Though our considerations in this section hold for all $n \geqslant 2$, we shall restrict ourselves to the case $n = 2$ to keep the exposition simple.

We start with a slightly weaker version of the classical theorem of L. Schwartz ([23]).

Theorem 2.1 (L. Schwartz's theorem on spectral analysis). *Let $\mathscr{U}$ be a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})$, which is closed under translations, then $\mathscr{U}$ contains an exponential function $e^{i\lambda x}$ for some $\lambda \in \mathbf{C}$.*

As pointed out in [1] an immediate corollary of the theorem is: If $\mathscr{U}$ is a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})_e$ which is closed under convolution against all $T \in \mathscr{E}'(\mathbf{R})_e$, then $\mathscr{U}$ contains a function of the form $\psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$.

We now introduce a family of functions on $\mathbf{R}^2$ which is central to spectral analysis of radial functions. For $\lambda \in \mathbf{C}$, define

$$\phi_\lambda(x) = \int_{|w|=1} e^{-i\lambda(x \,.\, w)} dw, \; x \in \mathbf{R}^2$$

where the integral is with respect to the normalised Lebesgue measure on the unit circle. Here $x \,.\, w$ is the usual inner product. It is immediate that ϕ_λ is a radial function for each $\lambda \in \mathbf{C}$. For $f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}$, we define a transform (sometimes called the Bessel transform):

$$\mathscr{G}f(\lambda) = \int_{\mathbf{R}^2} \phi_\lambda(x) f(x) dx\,, \quad \lambda \in \mathbf{C}\,.$$

We see that if $\hat{f}$ is the Fourier-Laplace transform of $f \in C_c^\infty(\mathbf{R}^2)$, i.e.,

$$\hat{f}(z_1, z_2) = \int_{\mathbf{R}^2} e^{-i(z \cdot x)} f(x) dx\,, \quad z = (z_1, z_2) \in \mathbf{C}^2\,,$$

then we have

$$\mathscr{G}f(\lambda) = \hat{f}(\lambda, 0)\,, \ \lambda \in \mathbf{C}\,, \ f \in C_c^\infty(\mathbf{R}^2)_{\mathrm{rad}}\,. \tag{2.1}$$

Both the transforms $\mathscr{G}$ defined above and the Fourier-Laplace transform have their obvious extension to $\mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$. We have for $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$,

$$\mathscr{G}T(\lambda) = T(\phi_\lambda)\,, \quad \lambda \in \mathbf{C}$$

$$\hat{T}(z_1, z_2) = T(e_z)\,, \quad z \in \mathbf{C}^2$$

where $e_z(x) = e^{-i(z_1 x_1 + z_2 x_2)} = e^{-i(z \cdot x)}$. We again have

$$\mathscr{G}T(\lambda) = \hat{T}(\lambda, 0)\,, \quad \lambda \in \mathbf{C}\,.$$

By applying the Paley-Wiener theorem we are able to obtain a description of the function space $\mathscr{X} = \{\mathscr{G}T : T \in C_c^\infty(\mathbf{R}^2)_{\mathrm{rad}}\}$.

LEMMA 2.2. *$\mathscr{X}$ is the space of even entire functions f on $\mathbf{C}$ such that for some constants, c, N and A (depending on f),*

$$|f(\lambda)| \leqslant C(1+|\lambda|)^N e^{A|\mathrm{Im}\,\lambda|}\,, \quad \lambda \in \mathbf{C}\,.$$

Proof. By the Paley-Wiener theorem an entire function $\phi = \hat{T}$ for some $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$ if and only if for some $C, N, A > 0$,

$$|\phi(z)| \leqslant C(1+|z|)^N e^{A|\mathrm{Im}\,z|}$$

and

$$\phi(z) = \phi(\sigma z)$$

for all $z = (z_1, z_2) \in \mathbf{C}^2$ and $\sigma \in SO(2, \mathbf{R})$ (here $\mathrm{Im}\,z = (\mathrm{Im}\,z_1, \mathrm{Im}\,z_2)$. The latter condition is equivalent to saying $\phi(z) = \phi(z')$ whenever $z_1^2 + z_2^2 = z_1'^2 + z_2'^2$. To see this, consider, for each $\alpha \in \mathbf{C}$,

$$M_\alpha = \{z : z_1^2 + z_2^2 = \alpha^2\}\,.$$

If $\alpha \neq 0$, M_α is a connected analytic submanifold of $\mathbf{C}^2$ of complex dimension 1 and $SO(2, \mathbf{R})(\alpha, 0, \ldots, 0)$ is a real submanifold of M_α of dimension 1 on which the analytic function ϕ is given to be constant. This forces ϕ to be a constant on M_α. A modification of the argument is necessary for $\alpha = 0$.

The lemma now follows from the simple observation that if $\lambda^2 = z_1^2 + z_2^2$, then $(\operatorname{Im} \lambda)^2 \leqslant (\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2$ and from the relation 2.1.

A straight-forward application of the one-dimensional Paley-Wiener theorem for even distributions of compact support will show that $\mathscr{X}$ is also equal to

$$\{\hat{T} : T \in \mathscr{E}'(\mathbf{R})_e\}\,.$$

This identification allows us to define the linear map Σ by

$$\Sigma : \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}} \to \mathscr{E}'(\mathbf{R}^2)_e$$
$$(\Sigma T)^\wedge(\lambda) = \mathscr{G}T(\lambda)\,, \quad \lambda \in \mathbf{C}\,.$$

Σ is one-to-one and onto. Moreover, we have the following description of the strong topology in $\mathscr{E}'(\mathbf{R}^n)$ (see [12], prop. 2.1): $T_n \to T$ if and only if (i) $\hat{T}_n \to \hat{T}$ uniformly on compact sets along with the derivatives and (ii) $\hat{T}_n$, $n \geqslant 1$ satisfy the uniform Paley-Wiener condition:

$$|\hat{T}_n(z)| \leqslant C(1+|z|)^N e^{A|\operatorname{Im} z|}\,, \quad z \in \mathbf{C}^n\,.$$

for some $C, N, A > 0$. This description coupled with the observation in the last step of the proof of Lemma 2.2 gives that Σ is a topological linear isomorphism between $\mathscr{E}'(\mathbf{R}^2)_{rad}$ and $\mathscr{E}'(\mathbf{R})_e$ preserving convolution.

On using the reflexivity of $\mathscr{E}(\mathbf{R})_e$ and $\mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$ we now get the map $\tilde{\Sigma}$:

$$\tilde{\Sigma} : \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}} \to \mathscr{E}(\mathbf{R})_e\,,$$
$$<\Sigma(T), \tilde{\Sigma}(f)> \;=\; <T, f> \quad T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}\,, f \in \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$$

where $< \cdot, \cdot >$ is the pairing of dual spaces.

We now have the following useful lemma:

LEMMA 2.3. *With the notation above, we have*

$$\Sigma(T) * \tilde{\Sigma}(f) = \tilde{\Sigma}(T * f)$$

for all $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$ *and* $f \in \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$, *where* $*$ *denotes the usual convolution on* $\mathbf{R}^2$.

Proof. Let $S \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$. Consider

$$<\Sigma(S), (\Sigma(T) * \tilde{\Sigma}(f))>

$$
\begin{aligned}
&= \Sigma(S) * \big(\Sigma(T)*\tilde{\Sigma}(f)\big)^{\vee}(0) \\
&\quad \big(\text{where } g^{\vee}(x) = g(-x),\, g \in \mathscr{E}(\mathbf{R}^2),\, x \in \mathbf{R}^2\big), \\
&= \Sigma(S) * \big((\tilde{\Sigma} f)^{\vee} * \Sigma(T)^{\vee}\big)(0) \\
&= \Sigma(S) * \big(\tilde{\Sigma}(f)*\Sigma(T)\big) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even} \\
&= \Sigma(S) * \big(\Sigma(T)*\tilde{\Sigma} f)\big)(0) \\
&= <\Sigma(S*T), \tilde{\Sigma} f> \\
&\quad (\text{using } \Sigma S * \Sigma T = \Sigma S * T) \\
&= <S*T, f> \\
&= S * T * f(0) \text{ as} f \text{ is even} \\
&= <S, T*f> .
\end{aligned}
$$

On the other hand,

$$<\Sigma(S), \tilde{\Sigma}(T*f)> = <S, T*f> .$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

THEOREM 2.4. *Let* $\mathscr{V}$ *be a closed nonzero subspace of* $\mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$ *such that for all* $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$ *and* $f \in \mathscr{V}, T * f \in \mathscr{V}$. *Then there exists* $\lambda \in \mathbf{C}$ *such that* $\phi_\lambda \in \mathscr{V}$.

Proof. Consider the closed and nontrivial subspace M of $\mathscr{E}(\mathbf{R})_e$ such that $\tilde{\Sigma}(\mathscr{V}) = M$. By Lemma 2.3, M is closed under convolution with elements $S \in \mathscr{E}'(\mathbf{R})_e$. By the remarks following Theorem 2.1 now, there exists $\lambda \in \mathbf{C}$ such that the functions $\Psi_\lambda \in M$, where $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$, $x \in \mathbf{R}$. A simple calculation now shows

$$<\phi_\lambda, f> = <\Psi_\lambda, \Sigma f> \quad f \in C_c^\infty(\mathbf{R}^2)_{\mathrm{rad}} \subseteq \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}} .$$

Thus $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$ and hence $\phi_\lambda \in \mathscr{V}$.

3. Pompeiu problem for the $M(2)$ action on $\mathbf{R}^2$

The Euclidean motion group $M(2)$ is the semidirect product of $\mathbf{R}^2$ with the rotation group $SO(2, \mathbf{R})$.