Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: EXCEPTIONAL POLYNOMIALS AND THE REDUCIBILITY OF

SUBSTITUTION POLYNOMIALS

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Kapitel: 2. The semi-factorable families

DOI: https://doi.org/10.5169/seals-57902

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certainly, as we shall see, an indecomposable EP f for which φ_f has a cubic factor lies in C_4 but whether this extends is unclear. More generally, in connection with EPs two questions naturally arise.

- (i) Are all indecomposable EPs over \mathbf{F}_q semi-factorable?
- (ii) Are all indecomposable semi-factorable EPs C-polynomials?

I would tentatively suggest that the answer to (ii) might be "yes" but hesitate to speculate on the answer to (i).

2. The semi-factorable families

The classes C_1 , C_2 and C_3 are described briefly (see [8], for example). More detail is given for C_4 .

 C_1 . Cyclic polynomials. These have the form $c_n(x) = x^n$, where $p \nmid n$. Obviously c_n is factorable and is an EP (or PP) if and only if g.c.d. (n, q-1) = 1. Trivially, of course, c_n is indecomposable over \mathbf{F}_q if and only if n is a prime $(\neq p)$.

 C_2 . Dickson polynomials. For any n(>1) with $p \nmid n$ and any $a(\neq 0)$ in \mathbf{F}_q , a typical member $g_n(x, a)$ has the shape

$$g_n(x, a) = \sum_{i=0}^{[n/2]} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

As in [13], over $\bar{\mathbf{F}}_a$ we have

(2.1)
$$\varphi_{g_n}(x, y) = (y - x) \prod_{i=1}^{\lfloor n/2 \rfloor} (y^2 - \alpha_i xy + x^2 + \beta_i^2 a) ,$$

where $\alpha_i = \zeta^i + \zeta^{-i}$, $\beta_i = \zeta^i - \zeta^{-i}$, ζ being a primitive *n*th root of unity in $\overline{\mathbf{F}}_q$. Since each of the quadratic factors in (2.1) is irreducible, g_n is not factorable. Yet it is semi-factorable. Set $R(x) = g_n(r_a(x), a)$, where $r_a(x) = x + ax^{-1}$. Then, by equation (7.8) of [8],

$$R(x) = r_{a^n}(c_n(x)) = x^n + (a/x)^n$$

and hence

$$\varphi_R(x, y) = \prod_{i=0}^{n-1} (y - \zeta^i x) (xy - \zeta^i a).$$

Thus R is factorable and g_n semi-factorable.

From (2.1) we can easily deduce the familiar facts that g_n is an EP or PP if and only if $(n, q^2 - 1) = 1$ while the identity

$$g_{n, m}(x, a) = g_{n}(g_{m}(x, a), a^{m})$$

((7.10) of [8]) yields the conclusion that $g_n(x, a)$ is indecomposable over \mathbf{F}_q if and only if n is a prime $(\neq p)$.

 C_3 . Linearised polynomials. These have degree $n=p^k(k\!\geqslant\!1)$, a typical specimen having the form

(2.2)
$$L(x) = \sum_{i=0}^{k} a_i x^{p^i},$$

where $a_0, ..., a_k \in \mathbf{F}_q$ with $a_0 a_k \neq 0$. Because $\varphi_L(x, y) = L(y - x)$, evidently L is factorable and is an EP (or PP) if and only if L has no non-zero roots in \mathbf{F}_q . Suppose that L is given by (2.1) but that, for some $s \geq 1$, $a_i = 0$ unless $s \mid i$. Then, for any $\alpha \in \mathbf{F}_{ps}$ and any $\beta \in \overline{\mathbf{F}}_q$, we have

(2.3)
$$L(\alpha x + \beta) = \alpha L(x) + \beta,$$

and we refer to L as a p^s -polynomial (cf. [8], § 3.4).

 C_4 . Sub-linearised polynomials. These polynomials (for whom a better title is requested) had their genesis in [1]. We construct a sub-linearised polynomial S(x) of degree $n=p^k(k\geq 1)$ as follows. Let L in C_3 be a p^s -polynomial of degree p^k and d(>1) be an integer such that $(p \not k) d \mid p^s - 1$. Then $L(x) = xM(x^d)$ for some $M(x) \in \mathbf{F}_q[x]$ and we set $S(x) = xM^d(x)$. Thus

$$S(x^d) = L^d(x)$$
,

or, equivalently,

$$S(c_d) = c_d(L).$$

The polynomial S as defined above will also be referred to as a (p^s, d) -polynomial. We note that, by (2.4) and Theorem 1.1 of [1], $S(c_d)$ is factorable and hence S is semi-factorable.

We remarked in [1] that a (p^s, d) -polynomial $S(x) = xM^d(x)$ for which M has no roots in \mathbf{F}_q is an EP provided $(d, p^{(s,t)} - 1) = 1$. In fact, the last condition is unnecessary and we state the definitive result as follows.

Theorem 2.1. Let $S(x) = xM^d(x)$ be a (p^s, d) -polynomial in $\mathbb{F}_q[x]$, where $d \mid p^s - 1$. Then

- (i) the irreducible factors of φ_S^* over \mathbf{F}_q all have degree d;
- (ii) S is an EP over \mathbf{F}_q if and only if M has no roots in \mathbf{F}_q .

Proof. (i) Since $d \mid p^s - 1$, then ζ , a primitive dth root of unity, lies in \mathbf{F}_{p^s} , and the non-zero roots of $L(x) \left(= xM(x^d) \right)$ can be arranged in the form $\{\zeta^j \gamma_h, j = 0, ..., d - 1, h = 1, ..., N\}$, where $N = \deg M = p^k - 1/d$ and $\{\gamma_h^d, h = 1, ..., N\}$ is the set of roots of M. By (2.3) and (2.4), we have

$$\varphi_{S}(x^{d}, y^{d}) = \varphi_{L^{d}}(x, y)
= \prod_{i=0}^{d-1} (L(y) - \zeta^{i}L(x))
= \prod_{i=0}^{d-1} L(y - \zeta^{i}x)
= (y^{d} - x^{d}) \prod_{i=0}^{d-1} \prod_{j=0}^{d-1} \prod_{h=1}^{N} (y - \zeta^{i}x - \zeta^{j}\gamma_{h})
= (y^{d} - x^{d}) \prod_{i=0}^{d-1} \prod_{j=0}^{d-1} \prod_{h=1}^{N} (\zeta^{i}y - \zeta^{j}x - \gamma_{h}).$$
(2.5)

Now, for any γ in $\overline{\mathbf{F}}_q$, it is clear that the polynomial

$$\prod_{i=0}^{d-1} \prod_{j=0}^{d-1} (\zeta^{j}y - \zeta^{j}x - \gamma)$$

lies in $\bar{\mathbf{F}}_q[x^d, y^d]$ and therefore may be written $P_\gamma(x^d, y^d)$, where $P_\gamma(x, y) \in \bar{\mathbf{F}}_q[x, y]$ has degree d (in both x and y). We claim that P_γ is irreducible. For suppose $P_\gamma(x, y)$ has a non-constant factor Q(x, y) in $\bar{\mathbf{F}}_q[x, y]$. Then $Q(x^d, y^d)$ must be divisible by $\zeta^i x - \zeta^j y - \gamma$ for some i and j with $0 \le i, j \le d - 1$. $Q(x^d, y^d)$, however, is invariant under $x \to \zeta^u x, y \to \zeta^v y$ for any u, v. It follows easily that $Q(x^d, y^d)$ is divisible by $P_\gamma(x^d, y^d)$ and we deduce that $Q = P_\gamma$, as required. Consequently, by (2.5),

$$\varphi_{S}^{*}(x, y) = \prod_{h=1}^{N} P_{\gamma_{h}}(x, y)$$

is the prime decomposition of ϕ_s^* over $\bar{\mathbf{F}}_q$ and (i) is proved.

(ii) Continuing with the same notation, we have

$$P_{\gamma}(x^{d}, y^{d}) = (-1)^{d} \prod_{i=0}^{d-1} (\gamma^{d} - (y - \zeta^{i}x)^{d})$$

= $(-1)^{d} \{ \gamma^{d^{2}} - d(y^{d} + (-x)^{d}) \gamma^{d(d-1)} + ... \}$.

It follows that, if γ^d is a root of M and $P_{\gamma}(x, y)$ lies in $\mathbf{F}_q[x, y]$, then both γ^{d^2} and $\gamma^{d(d-1)}$ are members of \mathbf{F}_q , whence $\gamma^d \in \mathbf{F}_q$. This means that S is an EP unless M has a root γ^d in \mathbf{F}_q . The converse is clear and the result follows.

3. Substitution polynomials with a quadratic factor

Throughout, let f(x) be an indecomposable polynomial in $\mathbf{F}_q[x]$ for which $\varphi_f(x, y)$ is divisible by an irreducible quadratic factor Q(x, y) in $\bar{\mathbf{F}}_q[x, y]$. Denote by Q^* the factor of φ_f , irreducible over \mathbf{F}_q itself, that is divisible by Q.

Lemma 3.1. Gal $Q^*(x, y)/\mathbf{F}_q(x)$ has order $\deg Q^*$ and so is regular as a permutation group on the roots of $Q^*(x, y)$ over $\mathbf{F}_q(x)$ (see [12], p. 8).

Proof. Let \mathbf{F}_{q^d} be the field generated over \mathbf{F}_q by the coefficients of Q (in $\overline{\mathbf{F}}_q$). Then $Q^* = \prod_{i=1}^d Q_i$, where $Q_1, ..., Q_d$ are the distinct conjugates of Q obtained by applying the d \mathbf{F}_q -automorphisms of \mathbf{F}_{q^d} to the coefficients of Q. Thus deg $Q^* = 2d$. But, evidently, the splitting field of Q^* over $\mathbf{F}_q(x)$ can be constructed by adjoining the splitting field of Q to \mathbf{F}_{q^d} . Its Galois group therefore has order 2d as required.

With Lemma 3.1 as a spur, we formulate some group theory in terms of polynomials (see [2]). For an indecomposable polynomial g(x) in $\mathbf{F}_q[x]$, $G = \operatorname{Gal}(g(y) - z/\mathbf{F}_q(z))$ is primitive. Moreover, the orbits of a point stabiliser G_x of G correspond to the irreducible factors of φ_g over \mathbf{F}_q ; in particular, when P(x, y) is such a factor of φ_g so also is P(y, x) and the associated orbits are "paired" (see [12], § 16). The following result is therefore a (slightly weakened) version of [12], Theorem 18.6.

LEMMA 3.2. With g and P as above, suppose that both $\operatorname{Gal} P(x, y)/\mathbf{F}_q(x)$ and $\operatorname{Gal} P(y, x)/\mathbf{F}_q(x)$ are regular. Then $\operatorname{Gal} \varphi_g(x, y)/\mathbf{F}_q(x) \cong \operatorname{Gal} P(x, y)/\mathbf{F}_q(x)$.

COROLLARY 3.3. With f and d as in Lemma 3.1, φ_f^* is a product over \mathbf{F}_q of irreducible polynomials of degree 2d, each of which is a product of irreducible quadratics over $\bar{\mathbf{F}}_q$. Furthermore, all these quadratics have a common splitting field over $\bar{\mathbf{F}}_q(x)$.