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SUBSTITUTION POLYNOMIALS
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certainly, as we shall see, an indecomposable EP f for which ϕ_f has a cubic factor lies in C_4 but whether this extends is unclear. More generally, in connection with EPs two questions naturally arise.

(i) Are all indecomposable EPs over \mathbf{F}_q semi-factorable?

(ii) Are all indecomposable semi-factorable EPs C -polynomials?

I would tentatively suggest that the answer to (ii) might be “yes” but hesitate to speculate on the answer to (i).

2. THE SEMI-FACTORABLE FAMILIES

The classes C_1 , C_2 and C_3 are described briefly (see [8], for example). More detail is given for C_4 .

C_1 . *Cyclic polynomials*. These have the form $c_n(x) = x^n$, where $p \nmid n$. Obviously c_n is factorable and is an EP (or PP) if and only if $\text{g.c.d.}(n, q-1) = 1$. Trivially, of course, c_n is indecomposable over \mathbf{F}_q if and only if n is a prime ($\neq p$).

C_2 . *Dickson polynomials*. For any $n(>1)$ with $p \nmid n$ and any $a(\neq 0)$ in \mathbf{F}_q , a typical member $g_n(x, a)$ has the shape

$$g_n(x, a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

As in [13], over $\bar{\mathbf{F}}_q$ we have

$$(2.1) \quad \phi_{g_n}(x, y) = (y-x) \prod_{i=1}^{\lfloor n/2 \rfloor} (y^2 - \alpha_i xy + x^2 + \beta_i^2 a),$$

where $\alpha_i = \zeta^i + \zeta^{-i}$, $\beta_i = \zeta^i - \zeta^{-i}$, ζ being a primitive n th root of unity in $\bar{\mathbf{F}}_q$. Since each of the quadratic factors in (2.1) is irreducible, g_n is not factorable. Yet it is semi-factorable. Set $R(x) = g_n(r_a(x), a)$, where $r_a(x) = x + ax^{-1}$. Then, by equation (7.8) of [8],

$$R(x) = r_{a^n}(c_n(x)) = x^n + (a/x)^n$$

and hence

$$\phi_R(x, y) = \prod_{i=0}^{n-1} (y - \zeta^i x) (xy - \zeta^i a).$$

Thus R is factorable and g_n semi-factorable.

From (2.1) we can easily deduce the familiar facts that g_n is an EP or PP if and only if $(n, q^2 - 1) = 1$ while the identity

$$g_{n,m}(x, a) = g_n(g_m(x, a), a^m)$$

((7.10) of [8]) yields the conclusion that $g_n(x, a)$ is indecomposable over \mathbf{F}_q if and only if n is a prime ($\neq p$).

C_3 . *Linearised polynomials.* These have degree $n = p^k (k \geq 1)$, a typical specimen having the form

$$(2.2) \quad L(x) = \sum_{i=0}^k a_i x^{p^i},$$

where $a_0, \dots, a_k \in \mathbf{F}_q$ with $a_0 a_k \neq 0$. Because $\phi_L(x, y) = L(y - x)$, evidently L is factorable and is an EP (or PP) if and only if L has no non-zero roots in \mathbf{F}_q . Suppose that L is given by (2.1) but that, for some $s \geq 1$, $a_i = 0$ unless $s \mid i$. Then, for any $\alpha \in \mathbf{F}_{p^s}$ and any $\beta \in \bar{\mathbf{F}}_q$, we have

$$(2.3) \quad L(\alpha x + \beta) = \alpha L(x) + \beta,$$

and we refer to L as a p^s -polynomial (cf. [8], § 3.4).

C_4 . *Sub-linearised polynomials.* These polynomials (for whom a better title is requested) had their genesis in [1]. We construct a sub-linearised polynomial $S(x)$ of degree $n = p^k (k \geq 1)$ as follows. Let L in C_3 be a p^s -polynomial of degree p^k and $d (> 1)$ be an integer such that $(p \nmid) d \mid p^s - 1$. Then $L(x) = xM(x^d)$ for some $M(x) \in \mathbf{F}_q[x]$ and we set $S(x) = xM^d(x)$. Thus

$$S(x^d) = L^d(x),$$

or, equivalently,

$$(2.4) \quad S(c_d) = c_d(L).$$

The polynomial S as defined above will also be referred to as a (p^s, d) -polynomial. We note that, by (2.4) and Theorem 1.1 of [1], $S(c_d)$ is factorable and hence S is semi-factorable.

We remarked in [1] that a (p^s, d) -polynomial $S(x) = xM^d(x)$ for which M has no roots in \mathbf{F}_q is an EP provided $(d, p^{(s,t)} - 1) = 1$. In fact, the last condition is unnecessary and we state the definitive result as follows.

THEOREM 2.1. *Let $S(x) = xM^d(x)$ be a (p^s, d) -polynomial in $\mathbf{F}_q[x]$, where $d \mid p^s - 1$. Then*

- (i) the irreducible factors of φ_S^* over \mathbf{F}_q all have degree d ;
(ii) S is an EP over \mathbf{F}_q if and only if M has no roots in \mathbf{F}_q .

Proof. (i) Since $d \mid p^s - 1$, then ζ , a primitive d th root of unity, lies in \mathbf{F}_{p^s} , and the non-zero roots of $L(x) (= xM(x^d))$ can be arranged in the form $\{\zeta^j \gamma_h, j=0, \dots, d-1, h=1, \dots, N\}$, where $N = \deg M = p^k - 1/d$ and $\{\gamma_h^d, h=1, \dots, N\}$ is the set of roots of M . By (2.3) and (2.4), we have

$$\begin{aligned}
 \varphi_S(x^d, y^d) &= \varphi_{L^d}(x, y) \\
 &= \prod_{i=0}^{d-1} (L(y) - \zeta^i L(x)) \\
 &= \prod_{i=0}^{d-1} L(y - \zeta^i x) \\
 &= (y^d - x^d) \prod_{i=0}^{d-1} \prod_{j=0}^{d-1} \prod_{h=1}^N (y - \zeta^i x - \zeta^j \gamma_h) \\
 (2.5) \quad &= (y^d - x^d) \prod_{i=0}^{d-1} \prod_{j=0}^{d-1} \prod_{h=1}^N (\zeta^i y - \zeta^j x - \gamma_h).
 \end{aligned}$$

Now, for any γ in $\bar{\mathbf{F}}_q$, it is clear that the polynomial

$$\prod_{i=0}^{d-1} \prod_{j=0}^{d-1} (\zeta^i y - \zeta^j x - \gamma)$$

lies in $\bar{\mathbf{F}}_q[x^d, y^d]$ and therefore may be written $P_\gamma(x^d, y^d)$, where $P_\gamma(x, y) \in \bar{\mathbf{F}}_q[x, y]$ has degree d (in both x and y). We claim that P_γ is irreducible. For suppose $P_\gamma(x, y)$ has a non-constant factor $Q(x, y)$ in $\bar{\mathbf{F}}_q[x, y]$. Then $Q(x^d, y^d)$ must be divisible by $\zeta^i x - \zeta^j y - \gamma$ for some i and j with $0 \leq i, j \leq d-1$. $Q(x^d, y^d)$, however, is invariant under $x \rightarrow \zeta^u x, y \rightarrow \zeta^v y$ for any u, v . It follows easily that $Q(x^d, y^d)$ is divisible by $P_\gamma(x^d, y^d)$ and we deduce that $Q = P_\gamma$, as required. Consequently, by (2.5),

$$\varphi_S^*(x, y) = \prod_{h=1}^N P_{\gamma_h}(x, y)$$

is the prime decomposition of φ_S^* over $\bar{\mathbf{F}}_q$ and (i) is proved.

- (ii) Continuing with the same notation, we have

$$\begin{aligned}
 P_\gamma(x^d, y^d) &= (-1)^d \prod_{i=0}^{d-1} (\gamma^d - (y - \zeta^i x)^d) \\
 &= (-1)^d \{ \gamma^{d^2} - d(y^d + (-x)^d) \gamma^{d(d-1)} + \dots \}.
 \end{aligned}$$

It follows that, if γ^d is a root of M and $P_\gamma(x, y)$ lies in $\mathbf{F}_q[x, y]$, then both γ^{d^2} and $\gamma^{d(d-1)}$ are members of \mathbf{F}_q , whence $\gamma^d \in \mathbf{F}_q$. This means that S is an EP unless M has a root γ^d in \mathbf{F}_q . The converse is clear and the result follows.

3. SUBSTITUTION POLYNOMIALS WITH A QUADRATIC FACTOR

Throughout, let $f(x)$ be an indecomposable polynomial in $\mathbf{F}_q[x]$ for which $\phi_f(x, y)$ is divisible by an irreducible quadratic factor $Q(x, y)$ in $\bar{\mathbf{F}}_q[x, y]$. Denote by Q^* the factor of ϕ_f , irreducible over \mathbf{F}_q itself, that is divisible by Q .

LEMMA 3.1. *Gal $Q^*(x, y)/\mathbf{F}_q(x)$ has order $\deg Q^*$ and so is regular as a permutation group on the roots of $Q^*(x, y)$ over $\mathbf{F}_q(x)$ (see [12], p. 8).*

Proof. Let \mathbf{F}_{q^d} be the field generated over \mathbf{F}_q by the coefficients of Q (in $\bar{\mathbf{F}}_q$). Then $Q^* = \prod_{i=1}^d Q_i$, where Q_1, \dots, Q_d are the distinct conjugates of Q obtained by applying the d \mathbf{F}_q -automorphisms of \mathbf{F}_{q^d} to the coefficients of Q . Thus $\deg Q^* = 2d$. But, evidently, the splitting field of Q^* over $\mathbf{F}_q(x)$ can be constructed by adjoining the splitting field of Q to \mathbf{F}_{q^d} . Its Galois group therefore has order $2d$ as required.

With Lemma 3.1 as a spur, we formulate some group theory in terms of polynomials (see [2]). For an indecomposable polynomial $g(x)$ in $\mathbf{F}_q[x]$, $G = \text{Gal}(g(y) - z/\mathbf{F}_q(z))$ is primitive. Moreover, the orbits of a point stabiliser G_x of G correspond to the irreducible factors of ϕ_g over \mathbf{F}_q ; in particular, when $P(x, y)$ is such a factor of ϕ_g so also is $P(y, x)$ and the associated orbits are "paired" (see [12], § 16). The following result is therefore a (slightly weakened) version of [12], Theorem 18.6.

LEMMA 3.2. *With g and P as above, suppose that both $\text{Gal } P(x, y)/\mathbf{F}_q(x)$ and $\text{Gal } P(y, x)/\mathbf{F}_q(x)$ are regular. Then $\text{Gal } \phi_g(x, y)/\mathbf{F}_q(x) \cong \text{Gal } P(x, y)/\mathbf{F}_q(x)$.*

COROLLARY 3.3. *With f and d as in Lemma 3.1, ϕ_f^* is a product over \mathbf{F}_q of irreducible polynomials of degree $2d$, each of which is a product of irreducible quadratics over $\bar{\mathbf{F}}_q$. Furthermore, all these quadratics have a common splitting field over $\bar{\mathbf{F}}_q(x)$.*