

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 36 (1990)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GAUSS SUMS AND THEIR PRIME FACTORIZATION
Autor: Brinkhuis, Jan
Kapitel: 3. The prime factorization of the Gauss sum: statement of the result
DOI: <https://doi.org/10.5169/seals-57901>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

corresponding to this isomorphism and let \mathfrak{P} be the prime in $\mathbf{Q}(pm)$ above \mathfrak{p} , so $\mathfrak{P}^{p-1} = \mathfrak{p}$, if we identify the prime ideal \mathfrak{p} of $\mathbf{Q}(m)$ with its extension to a fractional ideal of $\mathbf{Q}(pm)$. Thus we have the following congruence

$$(2.1) \quad \chi(x) \equiv x^{(p-1)/m} \pmod{\mathfrak{P}} \quad \text{for all } x \in \mathbf{F}_p^* .$$

Let $v_{\mathfrak{P}}$ be the valuation on $\mathbf{Q}(pm)$ corresponding to \mathfrak{P} . The number $\zeta_p - 1$ is a uniformizing element of $v_{\mathfrak{P}}$ in the sense that $v_{\mathfrak{P}}(\zeta_p - 1) = 1$. Moreover one has $v_{\mathfrak{P}}(p) = p - 1$. From the prime \mathfrak{P} we get the other primes in $\mathbf{Q}(pm)$ above p by Galois action: each prime in $\mathbf{Q}(pm)$ above p is equal to \mathfrak{P}^{τ} , the image of \mathfrak{P} under the Galois action of τ , for a unique $\tau \in \text{Gal}(\mathbf{Q}(m)/\mathbf{Q})$.

(2.2) In the same way we get from the prime \mathfrak{p} all the primes in $\mathbf{Q}(m)$ above p . However, in the last section of this paper, it will be more convenient to use a slightly different description of the primes in $\mathbf{Q}(m)$ above p . There we will not fix χ , as we do in the rest of the paper, but we will let it run over the $\phi(m)$ multiplicative characters on \mathbf{F}_p of order m . For each such χ we let $\mathfrak{p} = \mathfrak{p}(\chi)$ be the prime in $\mathbf{Q}(m)$ above p associated to χ in the way described above. Then $\mathfrak{p} = \mathfrak{p}(\chi)$ runs over the $\phi(m)$ primes in $\mathbf{Q}(m)$ above p .

3. THE PRIME FACTORIZATION OF THE GAUSS SUM:

STATEMENT OF THE RESULT

Before we state the outcome of the prime factorization of G we introduce some more notation. For each $i \in \mathbf{Z}$ with $0 < i < m$ and $(i, m) = 1$ we define the integer k_i to be the exponent of the prime $\mathfrak{P}^{\tau_i^{-1}}$ in the prime factorization of G in $\mathbf{Q}(pm)$ (it turns out that an inverse has to appear somewhere and this is a convenient place). Equivalently, k_i is the exponent of the prime \mathfrak{P} in the prime factorization of G^{τ_i} , that is,

$$(3.1) \quad k_i = v_{\mathfrak{P}}(G^{\tau_i}) .$$

Any given action of a group Γ on an algebraic number field F induces an action of the group Γ on $I(F)$, the group of fractional ideals in F . Now we proceed with it just as we did above with the action of Γ on the multiplicative group F^* : we denote the action of Γ on $I(F)$ by the

exponential notation, we extend it by \mathbf{Z} -linearity to an action of the group ring $\mathbf{Z}\Gamma$ on $I(F)$ and we denote this action also by the exponential notation. If moreover E is a subfield of F then we can view $I(E)$ as a subgroup of $I(F)$ by extension of fractional ideals; moreover if $\alpha \in I(E)$ with $\alpha = b^r$ for some $b \in I(F)$ and some $r \in \mathbf{N}$ and if $\lambda \in \mathbf{Q}\Gamma$ with $r\lambda \in \mathbf{Z}\Gamma$, then we make as usual the convention that the formal expression α^λ means the fractional ideal $b^{(r\lambda)}$ in F . We define the Stickelberger element θ in the group ring $\mathbf{Q}[\text{Gal}(\mathbf{Q}(m)/\mathbf{Q})]$ by

$$(3.2) \quad \theta = \sum_i \frac{i}{m} \tau_i^{-1}$$

where i runs over the positive integers $< m$ which are relatively prime to m . The formal expression \mathfrak{p}^θ denotes the ideal $\mathfrak{P}^{(p-1)\theta}$, by the convention made above for fractional exponents and by the relation $\mathfrak{p} = \mathfrak{P}^{p-1}$ between \mathfrak{p} and \mathfrak{P} .

Now we are ready to formulate the following result of Stickelberger on the Gauss sum G as defined in (1.1):

(3.3) THEOREM. *The prime factorization of the Gauss sum G is \mathfrak{p}^θ .*

(3.3) The statement of the theorem is clearly equivalent to the following one: only the primes in $\mathbf{Q}(pm)$ above p occur in the prime factorization of G , and their exponents in this factorization are as follows: for each positive integer $i < m$ which is relatively prime to m , the exponent of the prime $\mathfrak{P}^{\tau_i^{-1}}$ is $k_i = \frac{p-1}{m} i$.

4. A USEFUL LEMMA

In the proof of theorem (3.3) we will use a simple general lemma to determine the exponents in the prime factorization of the Gauss sum G . The aim of this section is to state and to prove this lemma. Let F be a field, v a discrete valuation on F , $F(v)$ the residue class field of v and π a uniformizing element of v , that is, $\pi \in F^*$ with $v(\pi) = 1$. An element $u \in F^*$ with $v(u) = 0$ will be called a v -unit. We define a homomorphism l from F^* to $\mathbf{Z} \times F(v)^*$ by sending each $\alpha \in F^*$ to the pair (k, r) consisting of the integer $k = v(\alpha)$ and the residue class r in $F(v)$ of the v -unit α/π^k . We call $l(\alpha)$