

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GAUSS SUMS AND THEIR PRIME FACTORIZATION  
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**Kapitel:** 2. The prime factorization of  $p$  in  $\mathbb{Q}(p^m)$   
**DOI:** <https://doi.org/10.5169/seals-57901>

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2. THE PRIME FACTORIZATION OF  $p$  IN  $\mathbf{Q}(pm)$ 

The next thing to do is to recall the prime factorization of the prime number  $p$  in the field  $\mathbf{Q}(pm)$  and to introduce a notation for the primes of  $\mathbf{Q}(pm)$  above  $p$  which is convenient for bookkeeping purposes. The prime number  $p$  ramifies completely in  $\mathbf{Q}(p)$ , in fact  $p \sim (\zeta_p - 1)^{p-1}$  where  $\sim$  denotes equality up to a factor which is an algebraic unit. The prime number  $p$  splits completely in  $\mathbf{Q}(m)$ , as  $p \equiv 1 \pmod{m}$ . These two facts determine by ramification theory the prime factorization of  $p$  in  $\mathbf{Q}(pm)$ : the prime number  $p$  splits completely in the extension  $\mathbf{Q}(m)/\mathbf{Q}$  and each prime in  $\mathbf{Q}(m)$  above  $p$  ramifies completely in the extension  $\mathbf{Q}(pm)/\mathbf{Q}(m)$ . This implies moreover that for each prime  $\mathfrak{Q}$  in  $\mathbf{Q}(pm)$  above  $p$  its residue field is  $\simeq \mathbf{F}_p$  and that the group  $\text{Gal}(\mathbf{Q}(pm)/\mathbf{Q}(m))$ , which we have identified with  $\text{Gal}(\mathbf{Q}(p)/\mathbf{Q})$ , is the inertia group of  $\mathfrak{Q}$  in the extension  $\mathbf{Q}(pm)/\mathbf{Q}$ , that is, it consists of the automorphisms of the field  $\mathbf{Q}(pm)$  which leave  $\mathfrak{Q}$  fixed and which moreover induce the trivial automorphism on the residue class field of  $\mathfrak{Q}$  (this last property is automatically satisfied as the residue class field is  $\simeq \mathbf{F}_p$  and so it has no non-trivial automorphisms).

Now we are going to give a more precise description of the primes in  $\mathbf{Q}(pm)$  above  $p$ . Let  $\phi$  be the Euler phi function defined on the natural numbers in one of the following, equivalent, ways:

- (i)  $\phi(n)$  is the number of positive integers  $\leq n$  which are relatively prime to  $n$ .
- (ii)  $\phi(n) = \#(\mathbf{Z}/n\mathbf{Z})^*$ .
- (iii)  $\phi(n) = [\mathbf{Q}(n) : \mathbf{Q}]$ .
- (iv)  $\phi(n)$  is the number of isomorphisms between two cyclic groups of order  $n$ .

For each field  $F$  and each  $n \in \mathbf{N}$  let  $\mu_n(F)$  be the group of  $n$ -th roots of unity in  $F$ ; this is in general a cyclic group of order dividing  $n$ . As  $m \mid p - 1$  the order of  $\mu_m(\mathbf{F}_p)$  is precisely  $m$ . The set of primes  $\mathfrak{q}$  in  $\mathbf{Q}(m)$  above  $p$  and the set of isomorphisms  $\psi$  from  $\mu_m(\overline{\mathbf{Q}})$  to  $\mu_m(\mathbf{F}_p)$  have both  $\phi(m)$  elements. In fact there is a canonical bijection between these two sets: let  $\mathfrak{q}$  correspond to  $\psi$  iff  $\zeta \equiv \psi(\zeta) \pmod{\mathfrak{q}}$  for all  $\zeta \in \mu_m(\overline{\mathbf{Q}})$ . Among those isomorphisms  $\psi$  we will now single one out. Let  $z$  be a generator of  $\mathbf{F}_p^*$ , then  $\chi(z)$  is a generator of  $\mu_m(\overline{\mathbf{Q}})$  and  $z^{(p-1)/m}$  is a generator of  $\mu_m(\mathbf{F}_p)$ . Therefore there is a unique isomorphism from  $\mu_m(\overline{\mathbf{Q}})$  to  $\mu_m(\mathbf{F}_p)$  which sends  $\chi(z)$  to  $z^{(p-1)/m}$ . It clearly sends  $\chi(x)$  to  $x^{(p-1)/m}$  for all  $x \in \mathbf{F}_p^*$ . This is the isomorphism which we single out. Let  $\mathfrak{p}$  be the prime in  $\mathbf{Q}(m)$  above  $p$

corresponding to this isomorphism and let  $\mathfrak{P}$  be the prime in  $\mathbf{Q}(pm)$  above  $\mathfrak{p}$ , so  $\mathfrak{P}^{p-1} = \mathfrak{p}$ , if we identify the prime ideal  $\mathfrak{p}$  of  $\mathbf{Q}(m)$  with its extension to a fractional ideal of  $\mathbf{Q}(pm)$ . Thus we have the following congruence

$$(2.1) \quad \chi(x) \equiv x^{(p-1)/m} \pmod{\mathfrak{P}} \quad \text{for all } x \in \mathbf{F}_p^* .$$

Let  $v_{\mathfrak{P}}$  be the valuation on  $\mathbf{Q}(pm)$  corresponding to  $\mathfrak{P}$ . The number  $\zeta_p - 1$  is a uniformizing element of  $v_{\mathfrak{P}}$  in the sense that  $v_{\mathfrak{P}}(\zeta_p - 1) = 1$ . Moreover one has  $v_{\mathfrak{P}}(p) = p - 1$ . From the prime  $\mathfrak{P}$  we get the other primes in  $\mathbf{Q}(pm)$  above  $p$  by Galois action: each prime in  $\mathbf{Q}(pm)$  above  $p$  is equal to  $\mathfrak{P}^{\tau}$ , the image of  $\mathfrak{P}$  under the Galois action of  $\tau$ , for a unique  $\tau \in \text{Gal}(\mathbf{Q}(m)/\mathbf{Q})$ .

(2.2) In the same way we get from the prime  $\mathfrak{p}$  all the primes in  $\mathbf{Q}(m)$  above  $p$ . However, in the last section of this paper, it will be more convenient to use a slightly different description of the primes in  $\mathbf{Q}(m)$  above  $p$ . There we will not fix  $\chi$ , as we do in the rest of the paper, but we will let it run over the  $\phi(m)$  multiplicative characters on  $\mathbf{F}_p$  of order  $m$ . For each such  $\chi$  we let  $\mathfrak{p} = \mathfrak{p}(\chi)$  be the prime in  $\mathbf{Q}(m)$  above  $p$  associated to  $\chi$  in the way described above. Then  $\mathfrak{p} = \mathfrak{p}(\chi)$  runs over the  $\phi(m)$  primes in  $\mathbf{Q}(m)$  above  $p$ .

### 3. THE PRIME FACTORIZATION OF THE GAUSS SUM:

#### STATEMENT OF THE RESULT

Before we state the outcome of the prime factorization of  $G$  we introduce some more notation. For each  $i \in \mathbf{Z}$  with  $0 < i < m$  and  $(i, m) = 1$  we define the integer  $k_i$  to be the exponent of the prime  $\mathfrak{P}^{\tau_i^{-1}}$  in the prime factorization of  $G$  in  $\mathbf{Q}(pm)$  (it turns out that an inverse has to appear somewhere and this is a convenient place). Equivalently,  $k_i$  is the exponent of the prime  $\mathfrak{P}$  in the prime factorization of  $G^{\tau_i}$ , that is,

$$(3.1) \quad k_i = v_{\mathfrak{P}}(G^{\tau_i}) .$$

Any given action of a group  $\Gamma$  on an algebraic number field  $F$  induces an action of the group  $\Gamma$  on  $I(F)$ , the group of fractional ideals in  $F$ . Now we proceed with it just as we did above with the action of  $\Gamma$  on the multiplicative group  $F^*$ : we denote the action of  $\Gamma$  on  $I(F)$  by the