Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

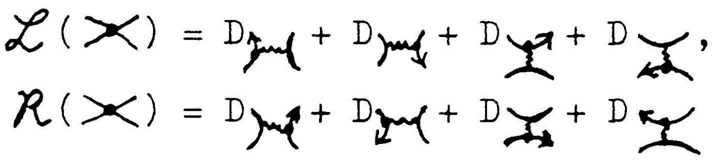
Artikel: STATE MODELS FOR LINK POLYNOMIALS

Autor: Kauffman, Louis H.

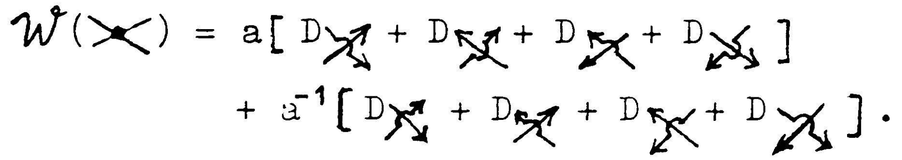
Kapitel: VI. The Conway Polynomial

DOI: https://doi.org/10.5169/seals-57900

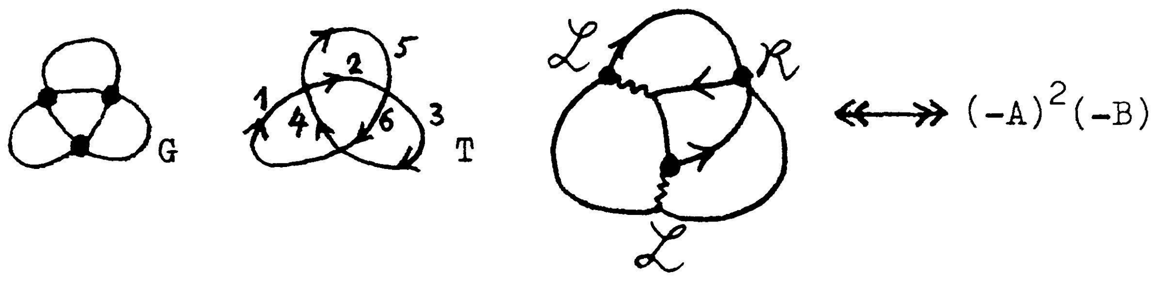
where



and



A state in this expansion is obtained by first splitting (in any way) the vertices of the given unoriented four-valent plane graph G. The vertex weights are then determined by the template, as illustrated below.



If $<G\,|\,S>$ denotes the product of vertex weights for a given state S, then the polynomial has the form

$$D_G = \sum_S <G\,|\,S>\ \mu^{|S|-1}, \quad \mu = 1 + (a - a^{-1})/(A - B)\,.$$

Proof of these formulas from the extension axioms follows just as in the Homfly case.

VI. The Conway Polynomial

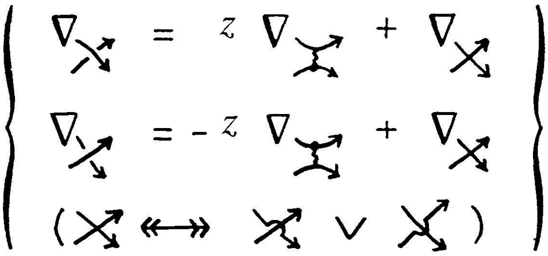
The skein models give a very elegant formulation of the Conway polynomial ([16], [41]) (compare [33])

$$\nabla_K(z) = R_K(z, 1)\,.$$

Specializing the formula for the skein model we have

$$\nabla_K(z) = \sum (-1)^{t_-(L)} z^{t(L)}$$

(summation over $L \in A(K. T)$, $| L | = 1$),



(Notation of section 2.)

Note that each state in this model has a single crossing circuit. Hence the template can be replaced by a single choice of base-point.

Writing

$$\nabla_K(z) = a_0(K) + a_1(K)z + a_2(K)z^2 + \cdots,$$

we have

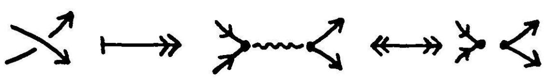
$$a_n(K) = \sum (-1)^{t_-(L)}$$

(summation over $L \in A(K, T)$, $| L | = 1$, $t(L) = n$).

Note that the second coefficient, $a_1(K)$, is the linking number of K when K is a 2-component link. The coefficients are generalized (self-)linking numbers.

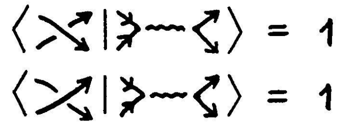
It is worth comparing this model with the model for the Conway polynomial given in *Formal Knot Theory* [42]. I shall refer to the latter model as the FKT model. The FKT model sums over all *Jordan Euler Trails* on the universe underlying the link diagram K. These trails result from splicing the diagram K at each crossing in either oriented

or non-oriented



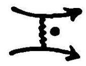
fashion. A choice of basepoint determines the vertex weights via the rules:

1. The unoriented splice has weight one.



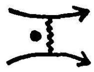
2. *Notation.*

Let



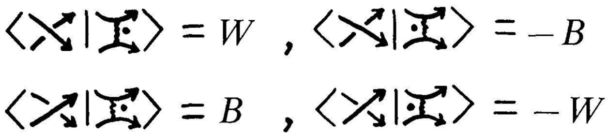
mean that the first passage through this site from the basepoint is *in the direction of the arrows.*

Let

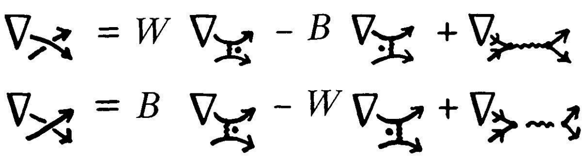


mean that the first passage through this site from the basepoint is *opposite to the direction of the arrows.*

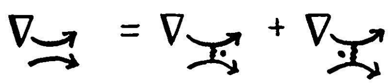
Then the weights are:



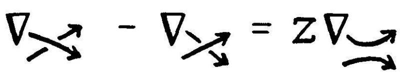
These vertex weights give the state expansion formulas:



with $z = W - B$, and $WB = 1$ (for topological invariance). Note that



is a tautology, and hence the Conway exchange identity



follows at once.

Definedness and properties of this model rest on a combinatorial result (the Clock Theorem [42]) from which it is straightforward to verify invariance under the Reidemeister moves. Furthermore, the model extends to a state model for the multi-variable Alexander-Conway polynomial (one variable for each component in a link). The FKT model is very closely related to Alexander's original approach to the polynomial via the Dehn presentation of the fundamental group of the link complement [6].

The FKT model has a number of intriguing features. It calculates a determinant of a generalized Alexander matrix. It is the low temperature limit of a generalized Potts model [57].

Is the FKT model a reformulation of the skein model for the Conway polynomial? There are a number of ways to try to generalize the FKT model to obtain a model of the Homfly polynomial. An answer to this question would shed light on the relationship of the FKT model and the Homfly polynomial. (And consequently on the relationship of the Homfly polynomial and the fundamental group of the link.)

VII. Yang-Baxter Models

I now turn to state models for specializations of the Homfly and Kauffman polynomials that arise from solutions to the Yang-Baxter Equation [10]. These models were devised by Vaughan Jones (Homfly) ([40]) and Volodja Turaev (Kauffman) ([93]). (See also the series of papers ([1], [2], [3], [4], [5], [64]) by Akutsu, Wadati and collaborators.) The reformulation of these models as given here is due to the author (compare [55], [58]).

The Yang-Baxter Equation arises in the study of two-dimensional statistical mechanics models [10] and also in the study of $1 + 1$ (1 space dimension, 1 time dimension) quantum field theory ([25], [100]). In the latter case, the motivation and relationship with knot theory is easiest to explain.

Regard a crossing in a universe (shadow of a link diagram) as a diagram for the interaction of two particles. Label the in-going and out-going lines of an oriented crossing with the "spins" of these particles. (Mathematically, spin is a generic term for a label chosen from an ordered index set $\mathscr{I}$. In applications it may denote the spin of a particle, or it may denote charge or some other intrinsic quantity.) The angle between the crossing segments can be regarded as an indicator of their relative momentum (rapidity). For each assignment of spins and each angle θ there will be a matrix element that, in the physical context, measures the amplitude (complex probability amplitude) for the process with these spins and rapidity.

The S matrix, $S_{cd}^{ab}(\theta)$, is said to be *factorized* if it satisfies the equations shown in Figure 8. This matrix equation is the Yang-Baxter Equation. Physically, it means that amplitudes for multi-particle interactions can be calculated from the two-particle scattering amplitude.