

# 6. The analytic proof

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that the image of  $t - s$  in  $J(C)$  is a constant section which completes the proof.  $\square$

6. THE ANALYTIC PROOF

In this section we will suppose  $K = \mathbf{C}$ .

a. *The Poincaré Lemma*

Suppose  $(\mathcal{S}, \nabla)$  is a sheaf on  $S^{an}$  with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathcal{S} \xrightarrow{\nabla} \Omega_{San}^1 \otimes \mathcal{S} \xrightarrow{\nabla} \Omega_{San}^2 \otimes \mathcal{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf  $\mathcal{S}^\nabla$ . Hence,

PROPOSITION 1.6.1.  $H^i(\mathcal{S}, \nabla)$  is naturally isomorphic to  $H^i(S, \mathcal{S}^\nabla)$ .

*Remark.* As in Proposition 1.1.1,  $H^1(\mathcal{S}, \nabla)$  is isomorphic to  $\text{Ext}(\mathcal{S}^\vee, \mathcal{O}_{San})$ . We can describe the isomorphism from  $H^1(\mathcal{S}, \nabla)$  to  $H^1(S, \mathcal{S}^\nabla)$  explicitly as follows: Let  $h$  be an element of  $H^1(\mathcal{S}, \nabla)$ . Let  $\mathcal{L}$  is a covering of  $S$  by open disks. Suppose  $\mathcal{E}$  is an extension of  $\mathcal{S}^\vee$  by  $\mathcal{O}_{San}$  corresponding to  $h$ . Then  $\mathcal{E}^\vee$  is an extension of  $\mathcal{O}_{San}$  by  $\mathcal{S}$ . For each  $U \in \mathcal{L}$ , there exists an  $s_U \in \mathcal{E}^\vee(U)^\nabla$  which maps to 1 in  $\mathcal{O}_{San}(U)$ . Then the image  $h$  in  $H^1(S, \mathcal{S}^\nabla)$  is the class of the cocycle  $\{(U, V) \rightarrow s_U - s_V\}$ .

Suppose,  $X$  is a smooth proper  $S$ -scheme and  $Z$  is a subscheme of  $X$  which is either empty or finite over  $S$ . We will define the Betti homology sheaf  $\mathcal{H}_i(X/S, Z, \mathbf{Z})$  on  $S^{an}$  as follows. If  $Z$  is smooth over  $S$ , we define  $\mathcal{H}_i(X/S, Z, \mathbf{Z})$  to be the sheaf associated to the presheaf

$$U \rightarrow H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z}),$$

(this latter group is the Betti homology of  $f^{-1}(U)$  relative to  $f^{-1}(U) \cap Z$ ). More generally, let  $S'$  be a non-empty affine open subset of  $S$  such that  $Z' = Z \times_S S'$  is étale over  $S'$ . Let  $X' = X \times_S S'$  and let  $\iota$  denote the inclusion morphisms  $X' \rightarrow X, Z' \rightarrow Z$  and  $S' \rightarrow S$ . We set

$$\mathcal{H}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathcal{H}_i(X'/S', Z', \mathbf{Z}).$$

This is independent of the choice of  $S'$ . We also set

$$\mathcal{H}_i(X/S, \mathbf{Z}) = \mathcal{H}_i(X/S, \emptyset, \mathbf{Z}) \text{ and } \mathcal{H}_1(X/S, Z, \mathbf{C}) = \mathcal{H}_1(X/S, Z, \mathbf{Z}) \otimes \underline{\mathbf{C}}.$$

Suppose  $s$  and  $t$  are two distinct sections of  $X/S$  and  $Z = s \cup t$ . Suppose  $S'$  is an affine open of  $S$  such that  $Z'$  is étale over  $S'$  in the notation of the previous paragraph. We have exact sequences

$$0 \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}) \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z}) \rightarrow \iota_* \mathcal{H}_0(Z'/S', \mathbf{Z}) \rightarrow \mathcal{H}_0(X/S, \mathbf{Z}) \rightarrow 0 .$$

and

$$0 \rightarrow \mathcal{H}_0(S'/S', \mathbf{Z}) \rightarrow \mathcal{H}_0(Z'/S', \mathbf{Z}) \rightarrow \mathcal{H}_0(X'/S', \mathbf{Z}) ,$$

where the first map is  $t_* - s_*$ . From which we derive the short exact sequence

$$0 \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}) \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z}) \rightarrow \underline{\mathbf{Z}} \rightarrow 0 .$$

since  $\iota_* \underline{\mathbf{Z}}|_{S'^{an}} \cong \underline{\mathbf{Z}}$ . In particular, if  $U$  is an open disk in  $S^{an}$ , we have an exact sequence

$$0 \rightarrow \mathcal{H}_1(X/S, \mathbf{Z})(U) \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z})(U) \rightarrow \mathbf{Z} \rightarrow 0$$

We define the Betti cohomology sheaf  $\mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C})$  in the same way and it is easy to see that  $\mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C}) \cong \text{Hom}(\mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{C}), \underline{\mathbf{C}})$ . Also, it is known that if  $Z$  is étale over  $S$  then  $\mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C}) \cong R_{f*}^1 \mathcal{I}_Z$  where  $\mathcal{I}_Z$  is the subsheaf of  $\underline{\mathbf{C}}$  whose sections vanish on  $Z$ .

Suppose  $X$  is proper over  $S$  with connected fibers. Let

$$(\mathcal{H}_{DR}^1(X/S, \mathbf{Z}), \nabla) = \mathcal{O}_{S^{an}} \otimes_{\mathcal{O}_S} (H_{DR}^1(X/S, \mathbf{Z}), \nabla) .$$

We claim, for  $Z \subseteq X$  finite over  $S$ .

$$(\mathcal{H}_{DR}^1(X/S, \mathbf{Z}), \nabla) \cong (\mathcal{O}_{S^{an}} \otimes \mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C}), d \otimes id)$$

This follows from the relative Poincaré lemma above on  $S'$  and hence on all of  $S$  since both sides are integrable connections. Hence,

LEMMA 1.6.2. *There is a natural isomorphism*

$$(\mathcal{H}_{DR}^1(X/S, \mathbf{Z})^\vee, \check{\nabla}) \cong (\mathcal{O}_{S^{an}} \otimes \mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C}), d \otimes id) .$$

*In particular*

$$H^i(\mathcal{H}_{DR}^1(X/S, \mathbf{Z})^\vee, \check{\nabla}) \cong H^i(S^{an}, \mathcal{H}^1(X/S, \mathbf{Z}, \mathbf{C})) .$$

We conclude, using this, Proposition 1.1.1 and GAGA that

THEOREM 1.6.3. *There exists a natural isomorphism*

$$\beta: \text{Ext}(H_{DR}^1(X/S), \mathcal{O}_S) \rightarrow H^1(S^{an}, \mathcal{H}^1(X/S, \mathbf{C})) .$$

b. *End of Analytic Proof*

Now suppose  $X$  is an Abelian scheme over  $S$ . We have an exact sequence of sheaves over  $S^{an}$ ,

$$0 \rightarrow \mathcal{H}_1(X/S, \mathbf{Z}) \rightarrow \mathcal{L}ie_{X^{an}/S^{an}} \rightarrow \underline{X^{an}} \rightarrow 0 .$$

From the corresponding long exact sequence of cohomology groups we obtain an exact sequence

$$\mathcal{L}ie_{X^{an}/S^{an}}(S^{an}) \rightarrow X^{an}(S^{an}) \xrightarrow{\delta} H^1(S^{an}, \mathcal{H}_1(X/S, \mathbf{Z})) .$$

We may describe  $\delta(s)$  as follows: Suppose  $e \neq s$ . Let  $Z = e \cup s$ . Then as  $f_*(\Omega_{X^{an}/S^{an}}^1)$  maps into  $\mathcal{H}_{DR}^1(X/S)$ ,  $\mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z})$  maps into

$$f_*(\Omega_{X^{an}/S^{an}}^1)^\vee = \mathcal{L}ie_{X^{an}/S^{an}}$$

so that the diagram

$$\begin{array}{ccc} \mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z}) & & \\ \uparrow & \searrow & \\ \mathcal{H}_1(X/S, \mathbf{Z}) & \rightarrow & \mathcal{L}ie_{X^{an}/S^{an}} \end{array}$$

commutes. Let  $\mathcal{U}$  be an ordered covering of  $S$  by open disks. For each  $U \in \mathcal{U}$  let  $\gamma_U \in \mathcal{H}_1(X/S, \mathbf{Z}, \mathbf{Z})(U)$  such that  $\gamma_U \rightarrow 1$  under the map  $\mathcal{H}_1(X/S, \mathbf{Z})(U) \rightarrow \mathbf{Z}$ . Then the image of  $\gamma_U$  in  $X(U)$  is  $s(U)$ . Hence  $\delta(s)$  is represented by the one cocycle  $\{(U, V) \rightarrow \gamma_U - \gamma_V\}$ .

Now, it follows from this and the remark after Proposition 1.6.1 that  $\beta \circ M$  is equal to the composition of  $\delta$  and the natural map

$$H^1(S^{an}, \mathcal{H}_1(X/S, \mathbf{Z})) \rightarrow H^1(S^{an}, \mathcal{H}_1(X/S, \mathbf{C})) \cong H^1(S^{an}, \mathcal{H}_1(X/S, \mathbf{Z})) \otimes \mathbf{C} .$$

Hence, if  $s \in X^{an}(S^{an})$ ,  $M(s) = 0$  iff there exists a positive integer  $n$  such that  $\delta(ns) = n\delta(s) = 0$ . Hence  $ns$  is in the image of  $\mathcal{L}ie_{X^{an}/S^{an}}(S^{an}) \rightarrow X^{an}(S^{an})$  and so is an infinitely divisible element of  $X^{an}(S^{an})$ .

Suppose  $s \in X(S)$ . We claim  $ns$  is an infinitely divisible element of  $X(S)$ . Let  $m$  be a positive integer. Let  $t \in X^{an}(S^{an})$  such that  $mt = ns$ . There exists a finite étale Galois covering  $\tilde{S}$  of  $S$  such that  $t \in X(\tilde{S})$ . If  $\sigma \in \text{Gal}(\tilde{S}/S)$ , then  $t^\sigma = t$  because  $t^\sigma(x) = t(\sigma^{-1}(x))$  for  $x \in \tilde{S}(\mathbf{C})$ . It follows that  $t \in X(S)$ . This establishes our claim.

Finally, it follows from the function field Mordell-Weil Theorem [LN] that the image of  $ns$  in  $X_{\mathbf{C}(S)}(\mathbf{C}(S))$  is a constant section  $X/S$ . Theorem 1.4.3 now follows immediately.  $\square$