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**FUNCTION FIELDS** 

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*Proof.* Let s and t be elements of A(S). Define the map  $g: A \to A$  by  $g = m \circ (id, t \circ f) (g(x) = x + t(f(x)))$ . Then  $g^*: H^1_{DR}(A/S) \to H^1_{DR}(A/S)$  is the identity so that  $g^*M(e, s) = M(e, s)$  on the one hand and  $g^*M(e, s) = M(t, s + t)$  by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$
 by Proposition 1.3.1.

Let  $(B, \tau)$  denote the K(S)/K trace of  $A_{K(S)}$  (see [L-AV]). In particular, B is an Abelian scheme over K and  $\tau: B \times \operatorname{spec}(K(S)) \to A_{K(S)}$  is a homomorphism. Since K has characteristic zero  $\tau$  is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of  $A_{K(S)}$  defined over over K. The morphism  $\tau$  extends uniquely to an S-morphism  $\bar{\tau}: B \times_K S \to A$ . It follows that B(K) maps naturally into A(S). We call the elements s of A(S) such that ns is in the image of B(K), the constant sections of A/S.

PROPOSITION 1.4.2. The kernel of M contains all constant sections of A/S.

*Proof.* Let s be a constant section of A/S. Then there exists a positive integer n such that  $ns = \bar{\tau} \circ (t \times id)$  where  $t \in B(K)$ . Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that  $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau} * M(t \times id) = 0$ . Since

$$\operatorname{Ext}(H^1_{DR}(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows.

We wish to prove the conserve of this proposition. I.e. we wish to prove:

Theorem 1.4.3. The kernel of M is precisely the group of all constant sections of A/S.

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

## 5. The algebraic proof

# a. Differentials with logarithmic singularities

(See [K] § 1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and

cross normally relative to T. Let W = X - Z and  $\tilde{Z}$  the disjoint union of the irreducible components of Z. Let  $(\Omega_{X/T}^{\cdot}(\text{Log}(Z)), d)$  denote the complex of differentials on X/T with logarithmic singularities along Z. (When T = K, we drop T from the notation.) When T has characteristic zero, which we will now assume, the i-th hypercohomology group of this complex is naturally isomorphic to  $H_{DR}^{i}(W/T)$ . We have a natural short exact sequence of complexes

$$0 \to \Omega_{X/T}^{\cdot} \to \Omega_{X/T}^{\cdot}(\text{Log}(Z)) \stackrel{\text{Res}}{\to} \Omega_{Z/T}^{\cdot}(-1) \to 0$$

From which, upon taking cohomology, we obtain the long exact sequence:

(5.1) 
$$0 \to H^{1}_{DR}(X/T) \to H^{1}_{DR}(W/T) \to H^{0}_{DR}(\tilde{Z}/T) \to H^{2}_{DR}(X/T) \\ \to H^{2}_{DR}(W/T) \to H^{2}_{DR}(\tilde{Z}/T)$$

In addition, we have a short exact sequence of complexes

$$0 \to \Omega_S^1 \otimes \Omega_{X/S}^{\cdot}(\operatorname{Log}(Z))(-1) \to \Omega_X^{\cdot}(\operatorname{Log}(Z)) \to \Omega_{X/S}^{\cdot}(\operatorname{Log}(Z)) \to 0.$$

The boundary maps in the long exact sequence of hypercohomology obtained from this short exact sequence are the Gauss-Manin connections  $\nabla: H^i_{DR}(W/S) \to \Omega^1_S \otimes H^i_{DR}(W/S)$ . Moreover the long exact sequence (5.1) is horizontal with respect to all the Gauss-Manin connections.

If D is any divisor on X, let  $\eta_T(D)$  denote the cohomology class of D in  $H^2_{DR}(X/T)$ . Recall ([H-DR; 7.7]), if  $\mathcal{L}$  is an ordered affine open cover of C and  $\{f_U\}$  is a Čech one-cochain with with coefficients in  $\mathcal{D}_A$  with respect to  $\mathcal{L}$  such that the divisor of  $f_U$  is the restriction of D to U, then  $\eta_T(D)$  is the cohomology class represented by the hyper one-cocycle  $\{0, \{d_{C/T} \text{Log}(f_{U,V})\}, 0\}$ , where  $f_{U,V} = f_U/f_V(U < V)$ . Suppose now that T is affine. Then  $H^0_{DR}(\tilde{Z}/T)$  is naturally isomorphic to the group of divisors on X supported on Z with coefficients in K[T].

LEMMA 1.5.1. Suppose D is a divisor on X supported on Z, then the image of D in  $H^2_{DR}(X/T)$  via the appropriate map in (5.1) is equal to  $\eta_T(D)$ .

*Proof.* This is essentially Proposition 7.6 of [H]. We carry out the proof in order to "straighten out" the sign.

Let  $\mathscr{L}$  be an affine open cover of X and  $\{f_U\}$  is a Čech one-chain with coefficients in  $\mathscr{D}_X$  with respect to  $\mathscr{L}$  such that the divisor of  $f_U$  is the restriction of D to U. Then,  $d\operatorname{Log}(f_U) \in \Omega^1_{X/T}(\operatorname{Log}(Z))(U)$  and  $\operatorname{Res}(d\operatorname{Log}(f_U))$  is the image of the image of D in  $H^0_{DR}(\tilde{Z}/T) \cong \mathscr{D}_{\tilde{Z}}(U)$ . It

follows that the image of D in  $H^2_{DR}(X/T)$  is the class of the hypercoboundary of  $(\{d\text{Log}(f_U)\},0)$  which is  $\eta_T(D)$  by definition.

By a properly semi-stable curve over S, we mean a curve over S such that the irreducible components of the closed fibers are smooth and cross normally. (The irreducible components do not have to be smooth if the curve is only semi-stable.)

COROLLARY 1.5.2. Suppose R is a smooth connected curve over a field K and X is a properly semi-stable curve over R smooth over K. Suppose U is a non-empty open subset of R and Y = R - U. Then the kernel of the natural map from  $H^2_{DR}(X)$  into  $H^2_{DR}(X_U)$  is generated by  $\{\eta(D)\}$  where D runs over the irreducible components of  $X_Y$ .

*Proof.* This follows from the lemma and the exact sequence (5.1), since the closed fibers of C/T are unions of smooth hypersurfaces of C which cross normally.  $\Box$ 

LEMMA 1.5.3. With notation as in the above corollary, if R is affine and X is smooth over R then the map from  $H^2_{DR}(X) \to H^2_{DR}(X_U)$  is an injection.

*Proof.* For a closed point x of R, let  $X_x$  denote the fiber above x. Since all the fibers of X over S are smooth, it follow from the corollary that the the kernel of the map  $H^2_{DR}(X) \to H^2_{DR}(X_U)$  is generated by  $\{\eta(X_x)\}$  where x runs over the closed points of Y. Now  $\eta(X_x)$  is the pull-back of  $\eta(x) \in H^2_{DR}(R)$ . As this latter group is zero, this proves the lemma.  $\square$ 

# b. End of algebraic proof

First by using the functoriality of M, Proposition 1.3.2, and the fact that every Abelian variety over S is the quotient of a Jacobian over S we may assume that A is the Jacobian of a smooth proper curve C over S. By Proposition 1.1.1 and the long exact sequence (2.3),  $\operatorname{Ext}(H_{DR}^1(C/S)^{\vee}, K[S])$  maps naturally into  $H_{DR}^2(C)$ . Moreover, since C is a proper smooth connected curve over S,  $H_{DR}^1(C/S)$  is canonically isomorphic to  $H_{DR}^1(C/S)^{\vee}$ . The fact we need to finish the proof is:

PROPOSITION 1.5.4. Let s and t be two elements of C(S). The class  $\eta(t-s)$  is equal to the image of M(s,t) in  $H^2_{DR}(C)$ .

By the previous lemma and the functoriality of  $\eta$  we may shrink S to suppose that  $s \cap t = \emptyset$ . To prove the proposition, we need the next lemma.

Suppose now that T = S,  $Z = s \cup t$  and X = C. Then the exact sequence (5.1) becomes:

(5.2) 
$$0 \to H^1_{DR}(C/S) \to H^1_{DR}(W/S) \to H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S) \to 0$$

Furthermore  $H^2_{DR}(C/S)$  is canonically isomorphic to K[S] with generator  $\eta_S(s) = \eta_S(t)$  and so the kernel of  $H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S)$  is a principal K[S] module with generator D = s - t. Using this generator, (5.2) yields an extension  $B_{s,t}$  of the connection (K[S],d) by  $(H^1_{DR}(C/S), \nabla)$ .

LEMMA 1.5.5. Identifying  $H^1_{DR}(C/S)$  with  $H^1_{DR}(C/S)^{\vee}$ , the extension  $B_{s,t}$  is isomorphic to the dual of  $E_{s,t}$ .

*Proof.* Regarding the complexes  $\Omega_{C/S,Z}$  and  $\Omega_{C/S}(\text{Log}(Z))$  as subcomplexes of  $\Omega_{W/S,Z}$  the wedge product gives a product from

$$\Omega_{C/S,Z}^{\cdot} \times \Omega_{C/S}^{\cdot}(\text{Log}(Z))$$

into  $\Omega_{C/S}$  which induces a pairing

$$(,): H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S) \to H^2_{DR}(C/S) \cong K[S]$$
.

This pairing is compatible with the exact sequences

$$0 \to H^0(C, \Omega^1_{C/S}) \to H^1_{DR}(C/S, Z) \to H^1(C, \Omega^0_{X/S}) \to 0$$
$$0 \to H^0(C, \Omega^1_{C/S}(\text{Log}(Z)) \to H^1_{DR}(W/S) \to H^1(C, \mathscr{D}_C) \to 0$$

arising from the Hodge to de Rham spectral sequences for hypercohomology (which degenerate). In other words, the image of  $H^0(C, \Omega^1_{C/S})$  in  $H^1_{DR}(C/S, Z)$  is perpendicular to the image of  $H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$  in  $H^1_{DR}(W/S)$  and if we identify  $\Omega^0_{X,Z}$  with  $\mathscr{O}_C(Z)$  and  $\Omega^1_{C/S}(\text{Log}(Z))$  with  $\Omega^1_{C/S}(-Z)$  the pairings induced on  $H^0(C, \Omega^1_{C/S}) \times H^1(C, \mathscr{O}_C)$  and on

$$H^1(C, \Omega_{X|Z}^0) \times H^0(C, \Omega_{C/S}^1(\text{Log}(Z)))$$

are the natural ones. Since these pairings are non-degenerate, it follows that the pairing on  $H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S)$  is non-degenerate.

It is also clear that the image of  $H_{DR}^0(Z/S) \cong K[Z]$  in  $H_{DR}^1(C/S, Z)$  is perpendicular to the image of  $H_{DR}^1(C/S)$  in  $H_{DR}^1(W/S)$  and that the pairing induced on  $H_{DR}^1(C/S) \times H_{DR}^1(C/S)$  is the natural one.

The lemma will follow from the following claim: Let  $\iota$  denote the map from K[Z] to  $H^1_{DR}(C/S, Z)$  and Res the map from  $H^1_{DR}(W/S)$  to K[Z]. Let  $T_{Z/S}$  denote the trace from K[Z] to K[S]. Suppose  $c \in K[Z]$  and  $\omega \in H^1_{DR}(W/S)$ . Then

$$(\iota(c), w) = - T_{Z/S}(c \operatorname{Res}(\omega)) .$$

Indeed, if s\*c = 0, t\*c = 1,  $Res_s(\omega) = 1$  and  $Res_t(\omega) = -1$  then  $-T_{Z/S}(cRes(\omega)) = 1$ .

To prove this claim we may shrink S. Hence, we may assume first that  $\{U,V\}(U < V)$  is an ordered affine open cover of C such that U = C - s and V = C - t, second that  $\iota(c)$  is represented by a hypercocycle of the form  $\partial(\{g_U,g_V\})$  where  $s*g_U = s*c$  and  $t*g_V = t*c$  and third, since the composition  $H^0(C,\Omega^1_{C/S}(\operatorname{Log}(Z)) \to H^1_{DR}(W/S) \to K[Z]$  is surjective, that w is in the image of  $H^0(C,\Omega^1_{C/S}(\operatorname{Log}(Z)))$ , i.e., w is represented by a hypercocycle of the form  $(\{\omega_U,\omega_V\},0)$  where  $\omega_U = \omega = \omega_V$  on  $U \cap V$  for some  $\omega \in H^0(C,\Omega^1_{C/S}(\operatorname{Log}(Z)))$ . It follows that  $(\iota(c),w)$  as an element of  $H^1_{DR}(C,\Omega^1_{C/S}) \cong H^2_{DR}(C/S)$  is represented by the cocycle  $\{v_{U,V}\}$  with  $v_{U,V} = (g_V - g_U)\omega$ . Since the image of this element in K[S] is

$$\operatorname{Res}_{s}(-g_{U}\omega) + -\operatorname{Res}_{t}(g_{V}\omega) = -(s*g_{U}\operatorname{Res}_{s}(\omega) + t*g_{V}\operatorname{Res}_{t}(\omega))$$
$$= -T_{T/S}(c\operatorname{Res}(\omega)).$$

this establishes the claim and the lemma.

# End of proof of Proposition 1.5.4

Consider the commutative diagram of complexes of sheaves with exact rows and columns

$$0 \qquad 0 \qquad 0 \qquad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \rightarrow \Omega_{S}^{1} \otimes \Omega_{C/S}^{:}(-1) \rightarrow \Omega_{C}^{:} \qquad \rightarrow \Omega_{C/S}^{:} \qquad \rightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \rightarrow \Omega_{S}^{1} \otimes \Omega_{C/S}^{:}(\text{Log}(Z))(-1) \rightarrow \Omega_{C}^{:}(\text{Log}(C)) \rightarrow \Omega_{C/S}^{:}(\text{Log}(Z) \rightarrow 0)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \rightarrow \Omega_{S}^{1} \otimes \Omega_{Z/S}^{:}(-2) \rightarrow \Omega_{Z}^{:}(-1) \rightarrow \Omega_{Z/S}^{:}(-1) \rightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \rightarrow \Omega_{S}^{1} \otimes \Omega_{Z/S}^{:}(-2) \rightarrow \Omega_{Z}^{:}(-1) \rightarrow 0$$

If we take hyper-cohomology of this diagram we obtain a commutative diagram

$$H^{1}_{DR}(C/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(C/S)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{1}_{DR}(W/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(W/S)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{0}_{DR}(Z) \rightarrow H^{0}_{DR}(Z/S)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{1}_{DR}(C/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(C/S) \rightarrow H^{2}_{DR}(C) \rightarrow H^{2}_{DR}(C/S)$$

with exact rows and columns in which the bottom row is part of the Leray long exact sequence. Let a be the element in  $H_{DR}^0(Z)$  corresponding to the divisor s-t. The image of a in  $H_{DR}^2(C)$  is  $\eta(s-t)$  by Lemma 1.5.1. On the other hand the image of a in  $H_{DR}^0(Z/S)$  is our chosen generator of the kernel of the map to  $H_{DR}^2(C/S)$ . In particular, it is the image of an element b of  $H_{DR}^1(W/S)$  and  $\nabla(b)$  is the image of an element c of  $\Omega_S^1 \otimes H_{DR}^1(C/S)$  whose image in  $H_{DR}^2(C/S)$  is the same as that of a by an elementary diagram chase. On the other hand, the image of c in  $H^1(H_{DR}^1(C/S), \nabla)$  is the class corresponding to the extension  $B_{s,t}$  by definition (see Proposition 1.1.1) which is, after identifying  $H_{DR}^1(C/S)$  with  $H_{DR}^1(C/S)^{\vee}$ , -M(s,t) by Lemma 1.5.1 and Lemma 1.5.3. Hence the image of M(s,t) in in  $H_{DR}^2(C)$  is  $-\eta(s-t) = \eta(t-s)$  as required.

Now we are in a position to prove the Theorem 1.4.3. We will suppose M(s,t)=0 which amounts to  $\eta_S(t-s)=0$  by the Proposition 1.5.1. Recall, that A is the Jacobian of C/S. Let d denote the divisor class of t-s in A(K(C)). We will show that the canonical height of d is zero. We may replace S by a finite étale cover and complete C to a properly semi-stable curve C over the completion S of S which is smooth over K. Let D be a  $\mathbb{Q}$ -rational divisor on C which is perpendicular (under the intersection pairing) to all the irreducible components of all the fibers of C/S and whose restriction to C is t-s. Such a divisor exists by the function field analogue of Theorem 1.3 of [Hr] (see also Theorem 5.1 (i) of [Ch]). It follows that the image of  $\eta(D)$  in  $H^2_{DR}(C)$  is  $\eta(t-s)=0$ . Corollary 1.5.2 implies that  $\eta(D)$  is in the span of  $\{\eta(Y)\}$  where Y runs over the irreducible component of the closed fibers above C-C. In particular,  $D \cdot D = 0$  using Theorem 7.8.2 of [H]. On the other hand,  $D \cdot D$  is -2 times the canonical height of d by the function field analogue of Theorem 5.1 of [Ch]. It now follows from Theorem 5.4.1 of [L],

that the image of t-s in J(C) is a constant section which completes the proof.  $\Box$ 

### 6. The analytic proof

In this section we will suppose  $K = \mathbb{C}$ .

## a. The Poincaré Lemma

Suppose  $(\mathcal{S}, \nabla)$  is a sheaf on  $S^{an}$  with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathscr{S} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \Omega^2_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf  $\mathcal{S}^{\nabla}$ . Hence,

PROPOSITION 1.6.1.  $H^i(\mathcal{S}, \nabla)$  is naturally isomorphic to  $H^i(S, \mathcal{S}^{\nabla})$ .

*Remark.* As in Proposition 1.1.1,  $H^1(\mathcal{S}, \nabla)$  is isomorphic to  $\operatorname{Ext}(\mathcal{S}^{\vee}, \mathcal{O}_{San})$ . We can describe the isomorphism from  $H^1(\mathcal{S}, \nabla)$  to  $H^1(S, \mathcal{S}^{\vee})$  explicitly as follows: Let h be an element of  $H^1(\mathcal{S}, \nabla)$ . Let  $\mathcal{S}$  is a covering of S by open disks. Suppose  $\mathcal{S}$  is an extension of  $\mathcal{S}^{\vee}$  by  $\mathcal{O}_{San}$  corresponding to h. Then  $\mathcal{S}^{\vee}$  is an extension of  $\mathcal{O}_{San}$  by  $\mathcal{S}$ . For each  $U \in \mathcal{S}$ , there exists an  $s_U \in \mathcal{S}^{\vee}(U)^{\nabla}$  which maps to 1 in  $\mathcal{O}_{San}(U)$ . Then the image h in  $H^1(S, \mathcal{S}^{\vee})$  is the class of the cocycle  $\{(U, V) \to s_U - s_V\}$ .

Suppose, X is a smooth proper S-scheme and Z is a subscheme of X which is either empty or finite over S. We will define the Betti homology sheaf  $\mathcal{M}_1(X/S, Z, \mathbf{Z})$  on  $S^{an}$  as follows. If Z is smooth over S, we define  $\mathcal{M}_i(X/S, Z, \mathbf{Z})$  to be the sheaf associated to the presheaf

$$U \rightarrow H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z})$$
,

(this latter group is the Betti homology of  $f^{-1}(U)$  relative to  $f^{-1}(U) \cap Z$ ). More generally, let S' be a non-empty affine open subset of S such that  $Z' = Z \times_S S'$  is étale over S'. Let  $X' = X \times_S S'$  and let  $\iota$  denote the inclusion morphisms  $X' \to X$ ,  $Z' \to Z$  and  $S' \to S$ . We set

$$\mathcal{L}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathcal{L}_i(X'/S', Z', \mathbf{Z})$$
.

This is independent of the choice of S'. We also set

 $\mathcal{M}_i(X/S, \mathbf{Z}) = \mathcal{M}_i(X/S, \emptyset, \mathbf{Z}) \text{ and } \mathcal{M}_1(X/S, Z, \mathbf{C}) = \mathcal{M}_1(X/S, Z, \mathbf{Z}) \otimes \mathbf{C}$ .