

5. The algebraic proof

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Proof. Let s and t be elements of $A(S)$. Define the map $g: A \rightarrow A$ by $g = m \circ (id, t \circ f)(g(x) = x + t(f(x)))$. Then $g^*: H_{DR}^1(A/S) \rightarrow H_{DR}^1(A/S)$ is the identity so that $g^*M(e, s) = M(e, s)$ on the one hand and $g^*M(e, s) = M(t, s + t)$ by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$

by Proposition 1.3.1. \square

Let (B, τ) denote the $K(S)/K$ trace of $A_{K(S)}$ (see [L-AV]). In particular, B is an Abelian scheme over K and $\tau: B \times \text{spec}(K(S)) \rightarrow A_{K(S)}$ is a homomorphism. Since K has characteristic zero τ is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of $A_{K(S)}$ defined over K . The morphism τ extends uniquely to an S -morphism $\bar{\tau}: B \times_K S \rightarrow A$. It follows that $B(K)$ maps naturally into $A(S)$. We call the elements s of $A(S)$ such that ns is in the image of $B(K)$, the constant sections of A/S .

PROPOSITION 1.4.2. *The kernel of M contains all constant sections of A/S .*

Proof. Let s be a constant section of A/S . Then there exists a positive integer n such that $ns = \bar{\tau} \circ (t \times id)$ where $t \in B(K)$. Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^*M(t \times id) = 0$. Since

$$\text{Ext}(H_{DR}^1(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows. \square

We wish to prove the converse of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. *The kernel of M is precisely the group of all constant sections of A/S .*

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

5. THE ALGEBRAIC PROOF

a. *Differentials with logarithmic singularities*

(See [K] §1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and

cross normally relative to T . Let $W = X - Z$ and \tilde{Z} the disjoint union of the irreducible components of Z . Let $(\Omega_{X/T}^\bullet(\text{Log}(Z)), d)$ denote the complex of differentials on X/T with logarithmic singularities along Z . (When $T = K$, we drop T from the notation.) When T has characteristic zero, which we will now assume, the i -th hypercohomology group of this complex is naturally isomorphic to $H_{DR}^i(W/T)$. We have a natural short exact sequence of complexes

$$0 \rightarrow \Omega_{X/T}^\bullet \rightarrow \Omega_{X/T}^\bullet(\text{Log}(Z)) \xrightarrow{\text{Res}} \Omega_{\tilde{Z}/T}^\bullet(-1) \rightarrow 0$$

From which, upon taking cohomology, we obtain the long exact sequence:

$$(5.1) \quad 0 \rightarrow H_{DR}^1(X/T) \rightarrow H_{DR}^1(W/T) \rightarrow H_{DR}^0(\tilde{Z}/T) \rightarrow H_{DR}^2(X/T) \\ \rightarrow H_{DR}^2(W/T) \rightarrow H_{DR}^2(\tilde{Z}/T)$$

In addition, we have a short exact sequence of complexes

$$0 \rightarrow \Omega_S^1 \otimes \Omega_{X/S}^\bullet(\text{Log}(Z))(-1) \rightarrow \Omega_X^\bullet(\text{Log}(Z)) \rightarrow \Omega_{X/S}^\bullet(\text{Log}(Z)) \rightarrow 0 .$$

The boundary maps in the long exact sequence of hypercohomology obtained from this short exact sequence are the Gauss-Manin connections $\nabla: H_{DR}^i(W/S) \rightarrow \Omega_S^1 \otimes H_{DR}^i(W/S)$. Moreover the long exact sequence (5.1) is horizontal with respect to all the Gauss-Manin connections.

If D is any divisor on X , let $\eta_T(D)$ denote the cohomology class of D in $H_{DR}^2(X/T)$. Recall ([H-DR; 7.7]), if \mathcal{L} is an ordered affine open cover of C and $\{f_U\}$ is a Čech one-cochain with coefficients in \mathcal{O}_A with respect to \mathcal{L} such that the divisor of f_U is the restriction of D to U , then $\eta_T(D)$ is the cohomology class represented by the hyper one-cocycle $(0, \{d_{C/T} \text{Log}(f_{U,V})\}, 0)$, where $f_{U,V} = f_U/f_V (U < V)$. Suppose now that T is affine. Then $H_{DR}^0(\tilde{Z}/T)$ is naturally isomorphic to the group of divisors on X supported on Z with coefficients in $K[T]$.

LEMMA 1.5.1. *Suppose D is a divisor on X supported on Z , then the image of D in $H_{DR}^2(X/T)$ via the appropriate map in (5.1) is equal to $\eta_T(D)$.*

Proof. This is essentially Proposition 7.6 of [H]. We carry out the proof in order to “straighten out” the sign.

Let \mathcal{L} be an affine open cover of X and $\{f_U\}$ is a Čech one-chain with coefficients in \mathcal{O}_X with respect to \mathcal{L} such that the divisor of f_U is the restriction of D to U . Then, $d\text{Log}(f_U) \in \Omega_{X/T}^1(\text{Log}(Z))(U)$ and $\text{Res}(d\text{Log}(f_U))$ is the image of the image of D in $H_{DR}^0(\tilde{Z}/T) \cong \mathcal{O}_{\tilde{Z}}(U)$. It

follows that the image of D in $H^2_{DR}(X/T)$ is the class of the hypercoboundary of $(\{d\text{Log}(f_U)\}, 0)$ which is $\eta_T(D)$ by definition. \square

By a properly semi-stable curve over S , we mean a curve over S such that the irreducible components of the closed fibers are smooth and cross normally. (The irreducible components do not have to be smooth if the curve is only semi-stable.)

COROLLARY 1.5.2. *Suppose R is a smooth connected curve over a field K and X is a properly semi-stable curve over R smooth over K . Suppose U is a non-empty open subset of R and $Y = R - U$. Then the kernel of the natural map from $H^2_{DR}(X)$ into $H^2_{DR}(X_U)$ is generated by $\{\eta(D)\}$ where D runs over the irreducible components of X_Y .*

Proof. This follows from the lemma and the exact sequence (5.1), since the closed fibers of C/T are unions of smooth hypersurfaces of C which cross normally. \square

LEMMA 1.5.3. *With notation as in the above corollary, if R is affine and X is smooth over R then the map from $H^2_{DR}(X) \rightarrow H^2_{DR}(X_U)$ is an injection.*

Proof. For a closed point x of R , let X_x denote the fiber above x . Since all the fibers of X over S are smooth, it follow from the corollary that the the kernel of the map $H^2_{DR}(X) \rightarrow H^2_{DR}(X_U)$ is generated by $\{\eta(X_x)\}$ where x runs over the closed points of Y . Now $\eta(X_x)$ is the pull-back of $\eta(x) \in H^2_{DR}(R)$. As this latter group is zero, this proves the lemma. \square

b. *End of algebraic proof*

First by using the functoriality of M , Proposition 1.3.2, and the fact that every Abelian variety over S is the quotient of a Jacobian over S we may assume that A is the Jacobian of a smooth proper curve C over S . By Proposition 1.1.1 and the long exact sequence (2.3), $\text{Ext}(H^1_{DR}(C/S)^\vee, K[S])$ maps naturally into $H^2_{DR}(C)$. Moreover, since C is a proper smooth connected curve over S , $H^1_{DR}(C/S)$ is canonically isomorphic to $H^1_{DR}(C/S)^\vee$. The fact we need to finish the proof is:

PROPOSITION 1.5.4. *Let s and t be two elements of $C(S)$. The class $\eta(t - s)$ is equal to the image of $M(s, t)$ in $H^2_{DR}(C)$.*

By the previous lemma and the functoriality of η we may shrink S to suppose that $s \cap t = \emptyset$. To prove the proposition, we need the next lemma.

Suppose now that $T = S, Z = s \cup t$ and $X = C$. Then the exact sequence (5.1) becomes:

$$(5.2) \quad 0 \rightarrow H_{DR}^1(C/S) \rightarrow H_{DR}^1(W/S) \rightarrow H_{DR}^0(\tilde{Z}/S) \rightarrow H_{DR}^2(C/S) \rightarrow 0$$

Furthermore $H_{DR}^2(C/S)$ is canonically isomorphic to $K[S]$ with generator $\eta_S(s) = \eta_S(t)$ and so the kernel of $H_{DR}^0(\tilde{Z}/S) \rightarrow H_{DR}^2(C/S)$ is a principal $K[S]$ module with generator $D = s - t$. Using this generator, (5.2) yields an extension $B_{s,t}$ of the connection $(K[S], d)$ by $(H_{DR}^1(C/S), \nabla)$.

LEMMA 1.5.5. *Identifying $H_{DR}^1(C/S)$ with $H_{DR}^1(C/S)^\vee$, the extension $B_{s,t}$ is isomorphic to the dual of $E_{s,t}$.*

Proof. Regarding the complexes $\Omega_{C/S,Z}^\bullet$ and $\Omega_{C/S}^\bullet(\text{Log}(Z))$ as subcomplexes of $\Omega_{W/S,Z}^\bullet$ the wedge product gives a product from

$$\Omega_{C/S,Z}^\bullet \times \Omega_{C/S}^\bullet(\text{Log}(Z))$$

into $\Omega_{C/S}^\bullet$ which induces a pairing

$$(\ , \): H_{DR}^1(C/S, Z) \times H_{DR}^1(W/S) \rightarrow H_{DR}^2(C/S) \cong K[S] .$$

This pairing is compatible with the exact sequences

$$\begin{aligned} 0 \rightarrow H^0(C, \Omega_{C/S}^1) \rightarrow H_{DR}^1(C/S, Z) \rightarrow H^1(C, \Omega_{X/S}^0) \rightarrow 0 \\ 0 \rightarrow H^0(C, \Omega_{C/S}^1(\text{Log}(Z))) \rightarrow H_{DR}^1(W/S) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow 0 \end{aligned}$$

arising from the Hodge to de Rham spectral sequences for hypercohomology (which degenerate). In other words, the image of $H^0(C, \Omega_{C/S}^1)$ in $H_{DR}^1(C/S, Z)$ is perpendicular to the image of $H^0(C, \Omega_{C/S}^1(\text{Log}(Z)))$ in $H_{DR}^1(W/S)$ and if we identify $\Omega_{X,Z}^0$ with $\mathcal{O}_C(Z)$ and $\Omega_{C/S}^1(\text{Log}(Z))$ with $\Omega_{C/S}^1(-Z)$ the pairings induced on $H^0(C, \Omega_{C/S}^1) \times H^1(C, \mathcal{O}_C)$ and on

$$H^1(C, \Omega_{X,Z}^0) \times H^0(C, \Omega_{C/S}^1(\text{Log}(Z)))$$

are the natural ones. Since these pairings are non-degenerate, it follows that the pairing on $H_{DR}^1(C/S, Z) \times H_{DR}^1(W/S)$ is non-degenerate.

It is also clear that the image of $H_{DR}^0(Z/S) \cong K[Z]$ in $H_{DR}^1(C/S, Z)$ is perpendicular to the image of $H_{DR}^1(C/S)$ in $H_{DR}^1(W/S)$ and that the pairing induced on $H_{DR}^1(C/S) \times H_{DR}^1(C/S)$ is the natural one.

The lemma will follow from the following claim: Let ι denote the map from $K[Z]$ to $H_{DR}^1(C/S, Z)$ and Res the map from $H_{DR}^1(W/S)$ to $K[Z]$. Let $T_{Z/S}$ denote the trace from $K[Z]$ to $K[S]$. Suppose $c \in K[Z]$ and $\omega \in H_{DR}^1(W/S)$. Then

$$(\iota(c), w) = - T_{Z/S}(c \text{Res}(\omega)) .$$

Indeed, if $s^*c = 0, t^*c = 1, \text{Res}_s(\omega) = 1$ and $\text{Res}_t(\omega) = -1$ then $- T_{Z/S}(c \text{Res}(\omega)) = 1$.

To prove this claim we may shrink S . Hence, we may assume first that $\{U, V\} (U < V)$ is an ordered affine open cover of C such that $U = C - s$ and $V = C - t$, second that $\iota(c)$ is represented by a hypercocycle of the form $\partial(\{g_U, g_V\})$ where $s^*g_U = s^*c$ and $t^*g_V = t^*c$ and third, since the composition $H^0(C, \Omega^1_{C/S}(\text{Log}(Z))) \rightarrow H^1_{DR}(W/S) \rightarrow K[Z]$ is surjective, that w is in the image of $H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$, i.e., w is represented by a hypercocycle of the form $(\{\omega_U, \omega_V\}, 0)$ where $\omega_U = \omega = \omega_V$ on $U \cap V$ for some $\omega \in H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$. It follows that $(\iota(c), w)$ as an element of $H^1_{DR}(C, \Omega^1_{C/S}) \cong H^2_{DR}(C/S)$ is represented by the cocycle $\{v_{U,V}\}$ with $v_{U,V} = (g_V - g_U)\omega$. Since the image of this element in $K[S]$ is

$$\begin{aligned} \text{Res}_s(-g_U\omega) + - \text{Res}_t(g_V\omega) &= - (s^*g_U \text{Res}_s(\omega) + t^*g_V \text{Res}_t(\omega)) \\ &= - T_{T/S}(c \text{Res}(\omega)) . \end{aligned}$$

this establishes the claim and the lemma. \square

End of proof of Proposition 1.5.4

Consider the commutative diagram of complexes of sheaves with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & \Omega^1_S \otimes \Omega^{\bullet}_{C/S}(-1) & \rightarrow & \Omega^{\bullet}_C & \rightarrow & \Omega^{\bullet}_{C/S} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \Omega^1_S \otimes \Omega^{\bullet}_{C/S}(\text{Log}(Z))(-1) & \rightarrow & \Omega^{\bullet}_C(\text{Log}(C)) & \rightarrow & \Omega^{\bullet}_{C/S}(\text{Log}(Z)) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \Omega^1_S \otimes \Omega^{\bullet}_{Z/S}(-2) & \rightarrow & \Omega^{\bullet}_Z(-1) & \rightarrow & \Omega^{\bullet}_{Z/S}(-1) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & . \end{array}$$

If we take hyper-cohomology of this diagram we obtain a commutative diagram

$$\begin{array}{ccc}
H_{DR}^1(C/S) & \xrightarrow{\nabla} & \Omega_S^1 \otimes H_{DR}^1(C/S) \\
\downarrow & & \downarrow \\
H_{DR}^1(W/S) & \xrightarrow{\nabla} & \Omega_S^1 \otimes H_{DR}^1(W/S) \\
\downarrow & & \\
H_{DR}^0(Z) & \rightarrow & H_{DR}^0(Z/S) \\
\downarrow & & \downarrow \\
H_{DR}^1(C/S) & \xrightarrow{\nabla} & \Omega_S^1 \otimes H_{DR}^1(C/S) \rightarrow H_{DR}^2(C) \rightarrow H_{DR}^2(C/S)
\end{array}$$

with exact rows and columns in which the bottom row is part of the Leray long exact sequence. Let a be the element in $H_{DR}^0(Z)$ corresponding to the divisor $s - t$. The image of a in $H_{DR}^2(C)$ is $\eta(s - t)$ by Lemma 1.5.1. On the other hand the image of a in $H_{DR}^0(Z/S)$ is our chosen generator of the kernel of the map to $H_{DR}^2(C/S)$. In particular, it is the image of an element b of $H_{DR}^1(W/S)$ and $\nabla(b)$ is the image of an element c of $\Omega_S^1 \otimes H_{DR}^1(C/S)$ whose image in $H_{DR}^2(C/S)$ is the same as that of a by an elementary diagram chase. On the other hand, the image of c in $H^1(H_{DR}^1(C/S), \nabla)$ is the class corresponding to the extension $B_{s,t}$ by definition (see Proposition 1.1.1) which is, after identifying $H_{DR}^1(C/S)$ with $H_{DR}^1(C/S)^\vee, -M(s, t)$ by Lemma 1.5.1 and Lemma 1.5.3. Hence the image of $M(s, t)$ in $H_{DR}^2(C)$ is $-\eta(s - t) = \eta(t - s)$ as required. \square

Now we are in a position to prove the Theorem 1.4.3. We will suppose $M(s, t) = 0$ which amounts to $\eta_s(t - s) = 0$ by the Proposition 1.5.1. Recall, that A is the Jacobian of C/S . Let d denote the divisor class of $t - s$ in $A(K(C))$. We will show that the canonical height of d is zero. We may replace S by a finite étale cover and complete C to a properly semi-stable curve \tilde{C} over the completion \tilde{S} of S which is smooth over K . Let D be a \mathbf{Q} -rational divisor on \tilde{C} which is perpendicular (under the intersection pairing) to all the irreducible components of all the fibers of \tilde{C}/\tilde{S} and whose restriction to C is $t - s$. Such a divisor exists by the function field analogue of Theorem 1.3 of [Hr] (see also Theorem 5.1 (i) of [Ch]). It follows that the image of $\eta(D)$ in $H_{DR}^2(C)$ is $\eta(t - s) = 0$. Corollary 1.5.2 implies that $\eta(D)$ is in the span of $\{\eta(Y)\}$ where Y runs over the irreducible component of the closed fibers above $\tilde{C} - C$. In particular, $D \cdot D = 0$ using Theorem 7.8.2 of [H]. On the other hand, $D \cdot D$ is -2 times the canonical height of d by the function field analogue of Theorem 5.1 of [Ch]. It now follows from Theorem 5.4.1 of [L],

that the image of $t - s$ in $J(C)$ is a constant section which completes the proof. \square

6. THE ANALYTIC PROOF

In this section we will suppose $K = \mathbf{C}$.

a. *The Poincaré Lemma*

Suppose (\mathcal{S}, ∇) is a sheaf on S^{an} with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathcal{S} \xrightarrow{\nabla} \Omega_{San}^1 \otimes \mathcal{S} \xrightarrow{\nabla} \Omega_{San}^2 \otimes \mathcal{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf \mathcal{S}^∇ . Hence,

PROPOSITION 1.6.1. $H^i(\mathcal{S}, \nabla)$ is naturally isomorphic to $H^i(S, \mathcal{S}^\nabla)$.

Remark. As in Proposition 1.1.1, $H^1(\mathcal{S}, \nabla)$ is isomorphic to $\text{Ext}(\mathcal{S}^\vee, \mathcal{O}_{San})$. We can describe the isomorphism from $H^1(\mathcal{S}, \nabla)$ to $H^1(S, \mathcal{S}^\nabla)$ explicitly as follows: Let h be an element of $H^1(\mathcal{S}, \nabla)$. Let \mathcal{L} is a covering of S by open disks. Suppose \mathcal{E} is an extension of \mathcal{S}^\vee by \mathcal{O}_{San} corresponding to h . Then \mathcal{E}^\vee is an extension of \mathcal{O}_{San} by \mathcal{S} . For each $U \in \mathcal{L}$, there exists an $s_U \in \mathcal{E}^\vee(U)^\nabla$ which maps to 1 in $\mathcal{O}_{San}(U)$. Then the image h in $H^1(S, \mathcal{S}^\nabla)$ is the class of the cocycle $\{(U, V) \rightarrow s_U - s_V\}$.

Suppose, X is a smooth proper S -scheme and Z is a subscheme of X which is either empty or finite over S . We will define the Betti homology sheaf $\mathcal{H}_1(X/S, Z, \mathbf{Z})$ on S^{an} as follows. If Z is smooth over S , we define $\mathcal{H}_i(X/S, Z, \mathbf{Z})$ to be the sheaf associated to the presheaf

$$U \rightarrow H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z}),$$

(this latter group is the Betti homology of $f^{-1}(U)$ relative to $f^{-1}(U) \cap Z$). More generally, let S' be a non-empty affine open subset of S such that $Z' = Z \times_S S'$ is étale over S' . Let $X' = X \times_S S'$ and let ι denote the inclusion morphisms $X' \rightarrow X, Z' \rightarrow Z$ and $S' \rightarrow S$. We set

$$\mathcal{H}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathcal{H}_i(X'/S', Z', \mathbf{Z}).$$

This is independent of the choice of S' . We also set

$$\mathcal{H}_i(X/S, \mathbf{Z}) = \mathcal{H}_i(X/S, \emptyset, \mathbf{Z}) \text{ and } \mathcal{H}_1(X/S, Z, \mathbf{C}) = \mathcal{H}_1(X/S, Z, \mathbf{Z}) \otimes \underline{\mathbf{C}}.$$