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connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that  $S$  is an affine curve over  $K$  and  $Z = \emptyset$ . Then the short exact sequence (2.2) becomes

$$0 \rightarrow f^* \Omega_S^1 \otimes \Omega_{X/S}^1(-1) \rightarrow \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^1 \rightarrow 0.$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$(2.3) \quad \dots \rightarrow H_{DR}^i(X/S) \xrightarrow{\nabla} \Omega_S^1 \otimes H_{DR}^i(X/S) \rightarrow H_{DR}^{i+1}(X) \rightarrow H_{DR}^{i+1}(X/S) \xrightarrow{\nabla} \dots$$

### 3. SECTIONS OF A FAMILY AND EXTENSIONS OF CONNECTIONS

Suppose now  $S$  is a smooth connected affine curve over a field  $K$  of characteristic zero and  $f: X \rightarrow S$  is a smooth proper morphism of schemes over  $K$ , with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose  $Z$  is a closed subscheme of  $X$  finite over  $S$ . Suppose the normalization  $n: \tilde{Z} \rightarrow Z$  of  $Z$  is smooth over  $S$ . After repeated blowing ups at closed points we find a scheme  $m: \tilde{X}' \rightarrow X$ , which contains  $\tilde{Z}$  and is such that the restriction of  $m$  to  $\tilde{Z}$  is  $n$ . Let  $\tilde{X}$  equal the complement in  $\tilde{X}'$  of the singular locus of  $\tilde{X}'/S$ . This locus is a closed subscheme of  $\tilde{X}'$  disjoint from  $\tilde{Z}$ . The long exact sequence 2.1 becomes

$$(3.1) \quad 0 \rightarrow K[S] \rightarrow K[\tilde{Z}] \rightarrow H_{DR}^1(\tilde{Z}/S, \tilde{Z}) \rightarrow H_{DR}^1(\tilde{X}/S) \rightarrow 0$$

Let  $H$  denote the pullback of  $H_{DR}^1(\tilde{X}/S, \tilde{Z})$  by means of the horizontal monomorphism from  $H_{DR}^1(X/S)$  into  $H_{DR}^1(\tilde{X}/S)$ . We claim that  $H$  is independent of the choice of  $\tilde{X}$ . Indeed, there exists a non-empty affine open subscheme  $S'$  of  $S$  such that the map from  $\tilde{X} \times_S S'$  to  $X' = :X \times_S S'$  is an isomorphism. If  $Z' = Z \times_S S'$ , then  $Z'$  is smooth over  $S'$  and it is easy to see that  $H \otimes K[S'] \cong H_{DR}^1(X'/S', Z')$ . Hence  $H$  is an extension of the connection  $H_{DR}^1(X'/S', Z')$  on  $S'$  to a connection on  $S$ . Since such an extension is unique if it exists, it follows that  $H$  is independent of the choice of  $\tilde{X}$  and so we set  $H_{DR}^1(X/S, Z) = H$ . We obtain from the previous exact sequence, a natural exact sequence

$$0 \rightarrow K[S] \rightarrow K[\tilde{Z}] \rightarrow H_{DR}^1(X/S, Z) \rightarrow H_{DR}^1(X/S) \rightarrow 0.$$

For a section  $s$  of  $X/S$ , we will also use  $s$  to denote the induced reduced closed subscheme  $s(S)$  of  $X$  when convenient. Now suppose  $s$  and  $t$  are two distinct sections of  $X/S$ . Let  $Z = s \cup t$ . Then  $\tilde{Z}$ , the normalization of  $Z$ , is just two disjoint copies of  $S$  and so is étale over  $S$ . (The sections  $s$  and  $t$

induce maps from  $S$  to  $\tilde{Z}$  which we denote by the same names.) The map  $t^* - s^*: K[\tilde{Z}] \rightarrow K[S]$  is horizontal, surjective and its kernel is the image of  $K[S]$  under the map in (3.1). Hence we obtain a horizontal exact sequence

$$0 \rightarrow K[S] \rightarrow H_{DR}^1(X/S, Z) \rightarrow H_{DR}^1(X/S) \rightarrow 0$$

and so an extension of  $H_{DR}^1(X/S)$  by the trivial connection. We let  $E(s, t)$  denote this extension if  $s \neq t$  and  $E(s, s)$  denote the trivial extension of  $H_{DR}^1(X/S)$  by  $K[S]$ . We call the class of  $E(s, t)$  in  $\text{Ext}(H_{DR}^1(X/S), K[S])$   $M(s, t)$ .

**PROPOSITION 1.3.1.** *Suppose  $r, s, t$  are sections of  $X/S$ . Then*

$$M(r, t) = M(r, s) + M(s, t).$$

*In particular,  $M(r, s) = -M(s, r)$ .*

*Proof.* If  $r, s$  and  $t$  are not distinct the proposition is obvious from the definitions. Therefore suppose that  $r, s$  and  $t$  are distinct. If  $T$  is a subset of  $\{r, s, t\}$  let  $Z_T = \bigcup_{u \in T} u$ . Either by replacing  $X$  by  $\tilde{X}$  or by shrinking  $S$  and using Corollary 1.1.3 we may assume that  $Z_{\{r, s, t\}}$  is étale over  $S$ . Let  $\mathcal{F}_T$  denote the complex  $\Omega_{X/S, Z_T}^\bullet$ . We set  $H(T) = H_{DR}^1(X/S, Z_T)$ . Then from the exact sequence of complexes

$$0 \rightarrow \mathcal{F}_{\{r, s, t\}} \rightarrow \mathcal{F}_{\{r, s\}} \otimes \mathcal{F}_{\{s, t\}} \rightarrow \mathcal{F}_{\{s\}} \rightarrow 0$$

(where the first map is the diagonal and the last is the difference) we obtain an exact sequence

$$0 \rightarrow H(r, s, t) \rightarrow H(r, s) \oplus H(s, t) \rightarrow H(s)$$

moreover,  $H(s) \cong H_{DR}^1(X/S)$  and the last map is the difference of the maps from  $H(r, s)$  and from  $H(s, t)$  to  $H_{DR}^1(X/S)$  (and is, in particular, a surjection).

Next from the exact sequence of complexes

$$0 \rightarrow \mathcal{F}_{\{r, s, t\}} \rightarrow \mathcal{F}_{\{r, t\}} \rightarrow \mathcal{S} \rightarrow 0$$

where  $\mathcal{S}$  is the complex  $(\mathcal{I}_{\{r, t\}} / \mathcal{I}_{\{r, s, t\}} \rightarrow 0 \rightarrow \dots) \cong (K[S] \rightarrow 0 \rightarrow \dots)$  we obtain an exact sequence

$$0 \rightarrow K[S] \rightarrow H(r, s, t) \rightarrow H(r, t) \rightarrow 0$$

Moreover the first map is the composition of the map from  $K[Z_{\{r, s, t\}}]$  into  $H(r, s, t)$  and the map  $h$  from  $K[S]$  into  $K[Z_{\{r, s, t\}}]$  characterized by  $r^*h(f) = t^*h(f) = 0$  and  $r^*h(f) = f$ . It follows from this that  $H(r, t)$  is the

Baer sum of  $H(r, s)$  and  $H(s, t)$ . Since all the maps discussed above are horizontal this statement is true on the level of connections as well. This proves the proposition.  $\square$

Suppose  $X'$  is a smooth scheme over  $S$  and  $g: X' \rightarrow X$  is an  $S$ -morphism. Then the natural map  $g^*: H_{DR}^1(X/S) \rightarrow H_{DR}^1(X/S)$  induces a natural map  $g^*: \text{Ext}(H_{DR}^1(X'/S), K[S]) \rightarrow \text{Ext}(H_{DR}^1(X/S), K[S])$ . By the naturality of all our constructions we have:

**PROPOSITION 1.3.2.** *Suppose  $X'/S$  has geometrically connected fibers and  $s$  and  $t$  are two sections of  $X'/S$ . Then*

$$M(g \circ s, g \circ t) = g^* M(s, t) .$$

Suppose  $X_0$  is a smooth connected scheme over  $K$  and  $X = S \times_K X_0$ . Then

$$(\Omega_{X/S}^\bullet, d_{X/S}) \cong K[S] \otimes (\Omega_{X_0/K}^\bullet, d_{X_0/K})$$

and so in particular,

$$H_{DR}^1(X/S) \cong K[S] \otimes H_{DR}^1(X_0/K)$$

and the Gauss-Manin connection

$$\nabla: H_{DR}^1(X/S) \rightarrow \Omega_S^1 \otimes_{K[S]} H_{DR}^1(X/S)$$

is  $(d, id)$ . If  $H = H_{DR}^1(X/S)$ , it follows from this that

$$\text{Ext}(H, K[S]) \cong H^1(\overset{\vee}{H}, \overset{\vee}{\nabla}) \cong \text{Hom}_K(H_{DR}^1(X_0/K), H_{DR}^1(S/K)) .$$

Explicitly, this last isomorphism can be described as follows:

$$\text{if } h \in \text{Hom}(H, \Omega_S^1) \cong \Omega_S^1 \otimes \overset{\vee}{H} ,$$

then  $h \bmod \overset{\vee}{\nabla} \overset{\vee}{H}$  goes to the map  $(\omega \in H_{DR}^1(X_0/K) \rightarrow h(1 \otimes \omega) \bmod dK[S])$ .

**PROPOSITION 1.3.3.** *Suppose  $X_0$  is a smooth connected scheme over  $K$  and  $X = S \times_K X_0$ . Suppose  $u$  and  $v$  are two morphisms from  $S$  to  $X_0$  and  $s = (id, u)$  and  $t = (id, v)$ . Then  $M(s, t)$  is  $v^* - u^*$  as an element of  $\text{Hom}_K(H_{DR}^1(X_0/K), H_{DR}^1(S/K))$ .*

*Proof.* We may suppose that  $s \cap t = \emptyset$ . Let  $Z = s \cup t$ . Suppose  $h: H_{DR}^1(X/S) \rightarrow H_{DR}^1(X/S, Z)$  is a section. Let  $(\{\omega_U\}, \{f_{U,V}\})$  be a one-hypercocycle for  $(\Omega_{X_0/K}^\bullet, d_{X_0/K})$  and  $[\omega]$  the image the class of  $1 \otimes (\{\omega_U\}, \{f_{U,V}\})$  in  $H_{DR}^1(X/S)$ . Then  $\nabla[\omega] = 0$ . We wish to compute  $\nabla h([\omega]) - h(\nabla[\omega]) = \nabla h([\omega])$ . We will abuse notation and identify  $\omega_U$  with  $1 \otimes \omega_U$  in  $\Omega_X^1(U)$  and  $f_{U,V}$  with  $1 \otimes f_{U,V}$  in  $\mathcal{O}_X(U \cap V)$ . Let  $\bar{\omega}_U$  denote the image of  $\omega_U$  in  $\Omega_{X/S}^1(U)$ . Then  $h([\omega])$  is the class of

$$(\{\bar{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain  $\{g_U\}$  with coefficients in  $\mathcal{O}_X$  such that

$$s^*f_{U,V} = u^*f_{U,V} = s^*(g_U - g_V) \quad \text{and} \quad t^*f_{U,V} = v^*f_{U,V} = t^*(g_U - g_V) .$$

Let  $\eta_U = \omega_U - dg_U$ . Now

$$s^*\eta_U - s^*\eta_V = s^*df_{U,V} - s^*d(g_U - g_V) = 0$$

by the conditions that  $\{g_U\}$  must satisfy and the fact that  $(\{\omega_U\}, \{f_{U,V}\})$  is a hypercocycle. Similarly,  $t^*\eta_U - t^*\eta_V = 0$ . Let  $\eta_s$  and  $\eta_t$  be the elements of  $\Omega_S^1$  determined by the cocycles  $\{s^*\eta_U\}$  and  $\{t^*\eta_U\}$  respectively.

Now to compute  $\nabla h([\omega])$  we must lift  $\bar{\omega}_U - d_{X/S}g_U$  to a section of  $\Omega_{X,Z}^1$ . Let  $e_{s,U}$  and  $e_{t,U}$  be elements of  $\mathcal{O}_X(U)$  such that  $s^*e_{s,U} = 1$ ,  $t^*e_{t,U} = 0$ ,  $t^*e_{t,U} = 1$  and  $s^*e_{t,U} = 0$ . These elements exist since  $Z$  is étale over  $S$ . Then  $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$  is such a lifting. To compute  $\nabla h([\omega])$  we must take the hyper-coboundary of  $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$ . It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0) .$$

The class of this hypercocycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1 \quad \text{in} \quad \Omega_S^1 \otimes H_{DR}^1(X/S, Z)$$

(recall that we've determined a map of  $K[S]$  into  $H_{DR}^1(X/S, Z)$ ). Hence  $\nabla h([\omega]) = \eta_t - \eta_s$ .

The proposition now follows from the fact that

$$(\{\eta_s + ds^*g_U\}, \{s^*g_U - s^*g_V\}) = u^*(\{\omega_U\}, \{f_{U,V}\})$$

and

$$(\{\eta_t + dt^*g_U\}, \{t^*g_U - t^*g_V\}) = v^*(\{\omega_U\}, \{f_{U,V}\}) . \quad \square$$

**COROLLARY 1.3.4.** *If, in the above,  $u$  and  $v$  are constant, then  $M(s, t) = 0$ .*

#### 4. ABELIAN SCHEMES

Suppose now that  $A$  is an Abelian scheme over  $S$ . Let  $m: A \times_S A \rightarrow A$  be the addition law and  $e$  the zero section. For  $s, t \in A(S)$ , let  $M(s) = M(e, s)$  and  $s + t = m(s, t)$ .

**THEOREM 1.4.1.** *The map  $M$  from  $A(S)$  to  $\text{Ext}(H_{DR}^1(A/S), K[S])$  is a homomorphism.*