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connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that S is an affine curve over K and $Z = \emptyset$. Then the short exact sequence (2.2) becomes

$$0 \to f^*\Omega^1_S \otimes \Omega^{\raisebox{3.5pt}{\text{\circle*{1.5}}}}_{X/S}(-1) \to \Omega^{\raisebox{3.5pt}{\text{\circle*{1.5}}}}_{X/S} \to \Omega^{\raisebox{3.5pt}{\text{\circle*{1.5}}}}_{X/S} \to 0 \ .$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$(2.3) \qquad \dots \to H^i_{DR}(X/S) \stackrel{\nabla}{\to} \Omega^1_S \otimes H^i_{DR}(X/S) \to H^{i+1}_{DR}(X) \to H^{i+1}_{DR}(X/S) \stackrel{\nabla}{\to} \dots$$

3. Sections of a family and extensions of connections

Suppose now S is a smooth connected affine curve over a field K of characteristic zero and $f: X \to S$ is a smooth proper morphism of schemes over K, with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose Z is a closed subscheme of X finite over S. Suppose the normalization $n: \tilde{Z} \to Z$ of Z is smooth over S. After repeated blowing ups at closed points we find a scheme $m: \tilde{X}' \to X$, which contains \tilde{Z} and is such that the restriction of m to \tilde{Z} is n. Let \tilde{X} equal the complement in \tilde{X}' of the singular locus of \tilde{X}'/S . This locus is a closed subscheme of \tilde{X}' disjoint from \tilde{Z} . The long exact sequence 2.1 becomes

$$(3.1) 0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(\tilde{Z}/S, \tilde{Z}) \to H^1_{DR}(\tilde{X}/S) \to 0$$

Let H denote the pullback of $H^1_{DR}(\tilde{X}/S,\tilde{Z})$ by means of the horizontal monomorphism from $H^1_{DR}(X/S)$ into $H^1_{DR}(\tilde{X}/S)$. We claim that H is independent of the choice of \tilde{X} . Indeed, there exists a non-empty affine open subscheme S' of S such that the map from $\tilde{X} \times_S S'$ to $X' = : X \times_S S'$ is an isomorphism. If $Z' = Z \times_S S'$, then Z' is smooth over S' and it is easy to see that $H \otimes K[S'] \cong H^1_{DR}(X'/S', Z')$. Hence H is an extension of the connection $H^1_{DR}(X'/S', Z')$ on S' to a connection on S. Since such an extension is unique if it exists, it follows that H is independent of the choice of \tilde{X} and so we set $H^1_{DR}(X/S, Z) = H$. We obtain from the previous exact sequence, a natural exact sequence

$$0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(X/S,Z) \to H^1_{DR}(X/S) \to 0 \ .$$

For a section s of X/S, we will also use s to denote the induced reduced closed subscheme s(S) of X when convenient. Now suppose s and t are two distinct sections of X/S. Let $Z = s \cup t$. Then \tilde{Z} , the normalization of Z, is just two disjoint copies of S and so is étale over S. (The sections s and t

induce maps from S to \tilde{Z} which we denote by the same names.) The map $t^* - s^* : K[\tilde{Z}] \to K[S]$ is horizontal, surjective and its kernel is the image of K[S] under the map in (3.1). Hence we obtain a horizontal exact sequence

$$0 \to K[S] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0$$

and so an extension of $H_{DR}^1(X/S)$ by the trivial connection. We let E(s,t) denote this extension if $s \neq t$ and E(s,s) denote the trivial extension of $H_{DR}^1(X/S)$ by K[S]. We call the class of E(s,t) in $\operatorname{Ext}(H_{DR}^1(X/S),K[S])$ M(s,t).

Proposition 1.3.1. Suppose r, s, t are sections of X/S. Then

$$M(r,t) = M(r,s) + M(s,t) .$$

In particular, M(r,s) = -M(s,r).

Proof. If r, s and t are not distinct the proposition is obvious from the definitions. Therefore suppose that r, s and t are distinct. If T is a subset of $\{r, s, t\}$ let $Z_T = \bigcup_{u \in T} u$. Either by replacing X by \tilde{X} or by shrinking S and using Corollary 1.1.3 we may assume that $Z_{\{r,s,t\}}$ is étale over S. Let \mathcal{F}_T denote the complex $\Omega^{\bullet}_{X/S,Z_T}$. We set $H(T) = H^1_{DR}(X/S,Z_T)$. Then from the exact sequence of complexes

$$0 \to \mathcal{F}_{\{r,s,t\}} \to \mathcal{F}_{\{r,s\}} \otimes \mathcal{F}_{\{s,t\}} \to \mathcal{F}_{\{s\}} \to 0$$

(where the first map is the diagonal and the last is the difference) we obtain an exact sequence

$$0 \rightarrow H(r, s, t) \rightarrow H(r, s) \oplus H(s, t) \rightarrow H(s)$$

moreover, $H(s) \cong H^1_{DR}(X/S)$ and the last map is the difference of the maps from H(r,s) and from H(s,t) to $H^1_{DR}(X/S)$ (and is, in particular, a surjection).

Next from the exact sequence of complexes

$$0 \to \mathcal{F}_{\{r,s,t\}} \to \mathcal{F}_{\{r,t\}} \to \mathcal{L} \to 0$$

where \mathcal{S} is the complex $(\mathcal{I}_{\{r,t\}}/\mathcal{I}_{\{r,s,t\}}\to 0\to ...)\cong (K[S]\to 0\to ...)$ we obtain an exact sequence

$$0 \to K[S] \to H(r,s,t) \to H(r,t) \to 0$$

Moreover the first map is the composition of the map from $K[Z_{\{r,s,t\}}]$ into H(r,s,t) and the map h from K[S] into $K[Z_{\{r,s,t\}}]$ characterized by $r^*h(f) = t^*h(f) = 0$ and $t^*h(f) = f$. It follows from this that H(r,t) is the

Baer sum of H(r, s) and H(s, t). Since all the maps discussed above are horizontal this statement is true on the level of connections as well. This proves the proposition. \square

Suppose X' is a smooth scheme over S and $g: X' \to X$ is an S-morphism. Then the natural map $g^*: H^1_{DR}(X/S) \to H^1_{DR}(X/S)$ induces a natural map $g^*: \operatorname{Ext}(H^1_{DR}(X'/S), K[S]) \to \operatorname{Ext}(H^1_{DR}(X/S), K[S])$. By the naturality of all our constructions we have:

PROPOSITION 1.3.2. Suppose X'/S has geometrically connected fibers and s and t are two sections of X'/S. Then

$$M(g \circ s, g \circ t) = g * M(s, t) .$$

Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Then

$$(\Omega_{X/S}, d_{X/S}) \cong K[S] \otimes (\Omega_{X_0/K}, d_{X_0/K})$$

and so in particular,

$$H^1_{DR}(X/S) \cong K[S] \otimes H^1_{DR}(X_0/K)$$

and the Gauss-Manin connection

$$\nabla: H^1_{DR}(X/S) \to \Omega^1_S \otimes_{K[S]} H^1_{DR}(X/S)$$

is (d, id). If $H = H_{DR}^1(X/S)$, it follows from this that

$$\operatorname{Ext}(H, K[S]) \cong H^{1}(\overset{\vee}{H}, \overset{\vee}{\nabla}) \cong \operatorname{Hom}_{K}(H^{1}_{DR}(X_{0}/K), H^{1}_{DR}(S/K)).$$

Explicitly, this last isomorphism can be described as follows:

if
$$h \in \text{Hom}(H, \Omega_s^1) \cong \Omega_s^1 \otimes \check{H}$$
,

then $h \mod \overset{\vee}{\nabla} \overset{\vee}{H}$ goes to the map $(\omega \in H^1_{DR}(X_0/K) \to h(1 \otimes \omega) \mod dK[S])$.

PROPOSITION 1.3.3. Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Suppose u and v are two morphisms from S to X_0 and s = (id, u) and t = (id, v). Then M(s, t) is $v^* - u^*$ as an element of $\operatorname{Hom}_K(H^1_{DR}(X_0/K), H^1_{DR}(S/K))$.

Proof. We may suppose that $s \cap t = \emptyset$. Let $Z = s \cup t$. Suppose $h: H^1_{DR}(X/S) \to H^1_{DR}(X/S, Z)$ is a section. Let $(\{\omega_U\}, \{f_{U,V}\})$ be a one-hypercocycle for $(\Omega^{\cdot}_{X_0/K}, d_{X_0/K})$ and $[\omega]$ the image the class of $1 \otimes (\{\omega_U\}, \{f_{U,V}\})$ in $H^1_{DR}(X/S)$. Then $\nabla[\omega] = 0$. We wish to compute $\nabla h([\omega]) - h(\nabla[\omega]) = \nabla h([\omega])$. We will abuse notation and identify ω_U with $1 \otimes \omega_U$ in $\Omega^1_X(U)$ and $f_{U,V}$ with $1 \otimes f_{U,V}$ in $\mathscr{P}_X(U \cap V)$. Let $\bar{\omega}_U$ denote the image of ω_U in $\Omega^1_{X/S}(U)$. Then $h([\omega])$ is the class of

$$(\{\tilde{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain $\{g_U\}$ with coefficients in \mathscr{O}_X such that

$$s^* f_{U,V} = u^* f_{U,V} = s^* (g_U - g_V)$$
 and $t^* f_{U,V} = v^* f_{U,V} = t^* (g_U - g_V)$.

Let $\eta_U = \omega_U - dg_U$. Now

$$s*\eta_U - s*\eta_V = s*df_{U,V} - s*d(g_U - g_V) = 0$$

by the conditions that $\{g_U\}$ must satisfy and the fact that $(\{\omega_U\}, \{f_{U,V}\})$ is a hypercocycle. Similarly, $t^*\eta_U - t^*\eta_V = 0$. Let η_s and η_t be the elements of Ω_S^1 determined by the cocycles $\{s^*\eta_U\}$ and $\{t^*\eta_U\}$ respectively.

Now to compute $\nabla h([\omega])$ we must lift $\bar{\omega}_U - d_{X/S}g_U$ to a section of $\Omega^1_{X,Z}$. Let $e_{s,U}$ and $e_{t,U}$ be elements of $\mathscr{O}_X(U)$ such that $s^*e_{s,U} = 1$ $t^*e_{t,U} = 0$, $t^*e_{t,U} = 1$ and $s^*e_{t,U} = 0$. These elements exist since Z is étale over S. Then $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$ is such a lifting. To compute $\nabla h([\omega])$ we must take the hyper-coboundary of $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$. It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0).$$

The class of this hypercocycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1$$
 in $\Omega_S^1 \otimes H_{DR}^1(X/S, Z)$

(recall that we've determined a map of K[S] into $H^1_{DR}(X/S, Z)$). Hence $\nabla h([\omega]) = \eta_t - \eta_s$.

The proposition now follows from the fact that

$$(\{\eta_s + ds^*g_U\}, \{s^*g_U - s^*g_V\}) = u^*(\{\omega_U\}, \{f_{U, V}\})$$

and

$$(\{\eta_t + dt^*g_U\}, \{t^*g_U - t^*g_V\}) = v^*(\{\omega_U\}, \{f_{U, V}\}). \qquad \Box$$

COROLLARY 1.3.4. If, in the above, u and v are constant, then M(s,t)=0.

4. ABELIAN SCHEMES

Suppose now that A is an Abelian scheme over S. Let $m: A \times_S A \to A$ be the addition law and e the zero section. For $s, t \in A(S)$, let M(s) = M(e, s) and s + t = m(s, t).

THEOREM 1.4.1. The map M from A(S) to $\operatorname{Ext}(H^1_{DR}(A/S), K[S])$ is a homomorphism.