where the sum runs over all intersections U of $i + 1$ distinct elements of $\mathcal{U}$. Let $\check{\partial}: C^i(\mathcal{U}, \mathcal{S}^j) \to C^{i+1}(\mathcal{U}, \mathcal{S}^j)$ be the Čech co-boundary. We also have boundaries $d: C^i(\mathcal{U}, \mathcal{S}^j) \to C^i(\mathcal{U}, \mathcal{S}^{j+1})$.

Now let

$$C^n(\mathcal{U}, \mathcal{S}^{\cdot}) = \oplus C^p(\mathcal{U}, \mathcal{S}^q)$$

where the sum runs over $p + q = n$. For $c \in C^n(\mathcal{U}, \mathcal{S}^{\cdot})$, we let $c^{p,q}$ denote its p, q-th component. The hyper-coboundary

$$\partial: C^n(\mathcal{U}, \mathcal{S}^{\cdot}) \to C^{n+1}(\mathcal{U}, \mathcal{S}^{\cdot})$$

is defined as follows: For $c \in C^n(\mathcal{U}, \mathcal{S}^{\cdot})$, we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1}\check{\partial}c^{p,q-1} .$$

Then the hypercohomology of $\mathcal{S}$ with respect to $\mathcal{C}$, $\mathbf{H}^{\cdot}(S, \mathcal{S}^{\cdot}, \mathcal{C})$, is defined to be $\mathrm{Ker}(\partial)/\mathrm{Image}(\partial)$ and $\mathbf{H}^{\cdot}(S, \mathcal{S}^{\cdot})$ is defined to be an appropriate limit of these groups over all ordered covers. In particular, if S is a scheme, $\mathcal{S}^{\cdot}$ is a complex of coherent sheaves and $\mathcal{C}$ is an affine open cover, then $\mathbf{H}^{\cdot}(S, \mathcal{S}^{\cdot})$ is naturally isomorphic to $\mathbf{H}^{\cdot}(S, \mathcal{S}^{\cdot}, \mathcal{C})$. If in addition S is affine $\mathbf{H}^{\cdot}(S, \mathcal{S}^{\cdot}) \cong H^{\cdot}(\Gamma(\mathcal{S}^{\cdot}))$.

1. EXTENSIONS OF CONNECTIONS

Let S be smooth connected scheme over a field K of characteristic zero. Suppose (H, ∇_H) and (G, ∇_G) are integrable connections on S. The set of isomorphism classes of integrable extensions of (H, ∇_H) by (G, ∇_G) forms a group under Baer sum which we will call $\mathrm{Ext}(H, G)$.

PROPOSITION 1.1.1. $\mathrm{Ext}(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$.

Proof. Since ∇_H is integrable, H is locally free. Let $\mathcal{C}$ be an ordered affine open cover of S such that $H(U)$ is a free $\mathcal{O}_S(U)$-module for each $U \in \mathcal{C}$. Suppose we have an extension

$$0 \to (G, \nabla_G) \to (E, \nabla) \to (H, \nabla_H) \to 0$$

of connections. Let $U \in \mathcal{C}$. Since $H(U)$ is free, there exists an $\mathcal{O}_S(U)$-module section $s_U: H(U) \to E(U)$. Now let $h_U = \nabla \circ s_U - s_U \circ \nabla_H$. We claim that h_U is an $\mathcal{O}_S(U)$-module homomorphism from $H(U)$ into $\Omega^1_S \otimes G(U)$, i.e. an element of $\mathrm{Hom}_{\mathcal{O}_S}(H, \Omega^1_S \otimes G)(U)$. Indeed, for $f \in \mathcal{O}_S(U)$ and $v \in H(U)$,

$$h_U(fv) = \nabla\big(s_U(fv)\big) - s_U\big(\nabla_H(fv)\big) = \nabla\big(fs_U(v)\big) - s_U(df \otimes v + f\nabla_H v)$$

$$= df \otimes s_U(v) - f\nabla\big(s_U(v)\big) - \big(df \otimes s_U(v) + fs_U(\nabla_H v)\big) = fh_U(v) \,.$$

Let $s_{U,V} = s_U - s_V \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)(U \cap V)$. We claim that $(\{h_U\}, \{s_{U,V}\})$ is a hyper one-cocycle for the complex $(\Omega_S^{\cdot} \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$. First it is clear that $\{s_{U,V}\}$ is a one-cocycle for the sheaf $\mathrm{Hom}_{\mathscr{O}_S}(H, G)$. Second

$$\nabla_G \circ s_{U,V} - s_{U,V} \circ \nabla_H = \nabla \circ (s_U - s_V) - (s_U - s_V) \circ \nabla_H = h_U - h_V \,.$$

Finally, since

$$\nabla \circ \nabla \circ s_U = \nabla \circ s_U \circ \nabla_H + \nabla \circ h_U = h_U \circ \nabla_H + \nabla_G \circ h_U = \nabla_{H,G}(h_U) \,,$$

(using Lemma 1.0.1) ∇ is integrable iff $\nabla_{H,G}(h) = 0$.

Moreover, suppose $\{s_U'\}$ is another collection of sections

$$s_U' : H(U) \to E(U), \quad h_U' = \nabla' \circ s_U' - s_U' \circ \nabla$$

and $s_{U,V}' = s_U' - s_V'$. Then $r_U = s_U' - s_U \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)$ and

$$h_U' = h + \nabla \circ r_U - r_U \circ \nabla_H = h + \nabla_G \circ r_U - r_U \circ \nabla_H = h + \nabla_{H,G}(r_U) \,.$$

And so $(\{h_U\}, \{s_{U,V}\}) - (\{h_U'\}, \{s_{U,V}'\})$ is the hyper-boundary of $\{r_U\}$. Thus we get a natural map from

$$\mathrm{Ext}(H, \mathscr{O}_X) \quad \text{into} \quad H^1(\mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \,.$$

It is easy to see that this map is a homomorphism.

We can make a map back as follows. Given a hyper-cocycle $(\{h_U\}, \{s_{U,V}\})$ for the complex $(\Omega_S^{\cdot} \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$, let E be the sheaf determined by the condition that $E(U) = G(U) \oplus H(U)$ with gluing data

$$(w, v) \to (w + s_{U,V}, v)$$

on $U \cap V$. We then put a connection ∇ on E by setting

$$\nabla(w, v) = \big(\nabla_G w + h_U(v), \nabla_H v\big)$$

for local sections w and v of G and H on U. One can check easily that E is an extension of H by G and that this construction gives the inverse to the map above. $\square$

COROLLARY 1.1.2. $\mathrm{Ext}(H, \mathscr{O}_S)$ is a K vector space and hence is uniquely divisible.

COROLLARY 1.1.3. Suppose S is affine and S' is a non-empty affine open of S. Then $\mathrm{Ext}(H, \mathscr{O}_S)$ injects into $\mathrm{Ext}(H \otimes \mathscr{O}_{S'}, \mathscr{O}_{S'})$.

We note that taking duals yields an isomorphism between $\operatorname{Ext}(G, H)$ and $\operatorname{Ext}(\check{H}, \check{G})$. Also, upon identifying $(\check{G})^{\vee}$ with G, $\check{\nabla}_{\check{G}}^{\vee} = \nabla_G$.

LEMMA 1.1.4. *The diagram*

$$
\begin{array}{ccc}
\operatorname{Ext}(H, G) & \to & H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \\
\downarrow & & \downarrow \\
\operatorname{Ext}(\check{G}, \check{H}) & \to & H^1(\check{H} \otimes G, \check{\nabla}_H \otimes \nabla_G)
\end{array}
$$

anti-commutes, where the horizontal arrows are the isomorphisms given by the proposition and the right vertical arrow is the evident one.

Proof. Since the assertion is local, we may suppose H and G are free. Suppose (E, ∇) is an extension of H by G and $s : H \to E$ is a section. Then $h = \nabla \circ s - s \circ \nabla_H$ is an element of $\operatorname{Hom}_{\mathscr{O}_S}(H, \Omega_S^1 \otimes G)$ which represents the image of the isomorphism class of E in

$$
H^1(\operatorname{Hom}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) .
$$

The image k of h in $\operatorname{Hom}_{\mathscr{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ is determined by

$$
k(w)(v) = w(h(v)) = w((\nabla \circ s - s \circ \nabla_H)(v))
$$

where v is a section of H and w is a section of $\check{G}$.

Now $(\check{E}, \check{\nabla})$ is an extension of $\check{G}$ by $\check{H}$ and the homomorphism t determined by

$$
t(w)(e) = w(e - s \circ \pi(e))
$$

is a section, where $\pi : E \to H$ is the projection, e is a section of E and w is a section of $\check{G}$. Hence, $g = \check{\nabla} \circ t - t \circ \nabla_{\check{G}}^{\vee}$ is an element of $\operatorname{Hom}_{\mathscr{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ which represents the image of the isomorphism class of $\check{E}$ in

$$
H^1(\operatorname{Hom}(\check{G}, \check{H}), \nabla_{\check{G}, \check{H}}^{\vee}) .
$$

Now

$$
g(w)(v) = (\check{\nabla} \circ t - t \circ \nabla_{\check{G}}^{\vee})(w)(e)
$$

where $e = s(v)$ and

$$
\check{\nabla} \circ t(w)(e) = d(w(e - s(\pi(e)) - w(\nabla(e) - s(\pi(\nabla(e)))))
$$

$$
= -w(\nabla \circ s(v) - s \circ \nabla_H(v)) = -k(w)(v)
$$

since $\pi(s(v)) = v$ and $\pi\nabla(e) = \nabla_H(\pi(e))$. The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e))) = 0 . \qquad \square$$

Suppose W is an $\mathscr{O}_S$ submodule of H. We let [W] denote the smallest subconnection of H containing W.

2. THE GAUSS-MANIN CONNECTION

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If $\mathscr{S}^{\boldsymbol{\cdot}}$ is a complex, $\mathscr{S}^{\boldsymbol{\cdot}}(k)$ will denote the complex obtained from $\mathscr{S}^{\boldsymbol{\cdot}}$ by setting $\mathscr{S}^i(k) = \mathscr{S}^{i+k}$. For any scheme Y over K will let $K[Y]$ denote $\Gamma(\mathscr{O}_Y)$.

Suppose S is a smooth connected affine scheme over K. Suppose $f: X \to S$ is a smooth morphism, Z is a closed subscheme of X, smooth over S. Suppose T is either $\mathrm{Spec}(K)$ or S. Then we define the subcomplex $\Omega_{X/T,Z}^{\boldsymbol{\cdot}}$ of $\Omega_{X/T}^{\boldsymbol{\cdot}}$ by the exactness of the sequence.

$$0 \to \Omega_{X/T,Z}^{\boldsymbol{\cdot}} \to \Omega_{X/T}^{\boldsymbol{\cdot}} \to \Omega_{Z/T}^{\boldsymbol{\cdot}} \to 0 .$$

When $T = \mathrm{Spec}(K)$ we drop it from the notation. It follows that $\Omega_{X/S,Z}^i = \Omega_{X/S}^i$ for $i > \dim_S Z$. Note that $\Omega_{X,Z}^0 = \Omega_{X/S,Z}^0$ is the sheaf of ideals of Z on X. We define $H_{DR}^i(X/S, Z)$ to be the i-th hypercohomology group of the complex $\Omega_{X/S,Z}^{\boldsymbol{\cdot}}$. We set $H_{DR}^i(X/S) = H_{DR}^i(X/S, \varnothing)$. If X is affine, then $H_{DR}^i(X/S, Z)$ is the i-th cohomology group of the complex of $K[S]$ modules $\Gamma(\Omega_{X/S,Z}^{\boldsymbol{\cdot}})$. If X is affine, K has characteristic zero and U is a dense open subscheme of X then the natural map from $H_{DR}^i(X/S, Z)$ to $H_{DR}^i(U/S, U \cap Z)$ is an injection.

From the last short exact sequence with $T = S$, we obtain a long exact sequence

$$(2.1) \qquad \ldots \to H_{DR}^{i-1}(Z/S) \to H_{DR}^i(X/S, Z) \to H_{DR}^i(X/S) \to \ldots$$

The Gauss-Manin connection $\nabla: H_{DR}^i(X/S, Z) \to \Omega_S^1 \otimes H_{DR}^i(X/S, Z)$ is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

$$(2.2) \quad 0 \to f^*\Omega_S^1 \otimes \Omega_{X/S,Z}^{\boldsymbol{\cdot}}(-1) \to \Omega_{X/S,Z}^{\boldsymbol{\cdot}}/f^*\Omega_S^2 \otimes \Omega_X^{\boldsymbol{\cdot}}(-2) \to \Omega_{X/S,Z}^{\boldsymbol{\cdot}} \to 0$$

(which is exact because X and Z are smooth over S). It is an integrable connection. If K has characteristic zero and f is surjective and has geometrically connected fibers, then $H_{DR}^0(X/S) = K[S]$ and the Gauss-Manin