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MANIN'S PROOF OF THE MORDELL CONJECTURE OVER FUNCTION FIELDS

by Robert F. COLEMAN

In the process of translating Manin's proof of Mordell's conjecture over function fields into modern language we found a gap. The arguments in [M] do not suffice to prove Manin's Theorem of the Kernel. We were able to fill this gap by using those arguments to prove a weaker theorem (Theorem 1.4.3 below) and combining this with the function field analogue of Siegel's Theorem and Manin's ideas to complete the proof of Function Field Mordell. More recently, Chai [C] (see also the Appendix, below) has applied Deligne's Theorem on the semi-simplicity of the action of the monodromy group to deduce Manin's Theorem of the Kernel as reformulated below from the weaker theorem mentioned above. I believe that all this is testimony to the power and depth of Manin's intuition. We were also able to make Manin's analytic proof completely algebraic. Manin has kindly verified that the corrections discussed herein are necessary and apt (see letter to Izvestia...)

In light of the above and because of the ground braking nature of the work we believe that Manin's paper "Rational Points of Algebraic Curves over Function Fields" merits a clear modern treatment. We attempt to give one below.

I. THE THEOREM OF THE KERNEL

0. REVIEW OF CONNECTIONS AND HYPERCOHOMOLOGY

(See also [D-1].) Let S be smooth connected scheme over a ring K. Let $\mathcal{O}_S$ denote the structure sheaf of S, Ω_S^p the sheaf of p-forms on S over K and d the exterior derivation from Ω_S^p to Ω_S^{p+1}. Let $\mathcal{S}$ be a coherent sheaf on S. A connection on $\mathcal{S}$ over K is a K-linear homomorphism $\nabla : \mathcal{S} \to \Omega_S^1 \otimes \mathcal{S}$ satisfying the Leibnitz rule

We are indebted to Arthur Ogus for many helpful and stimulating discussions.

$$\nabla(fs) = df \otimes s + f\nabla(s) .$$

for f a local section of $\mathcal{O}_S$ and s a local section of $\mathcal{S}$. We will also say that $(\mathcal{S}, \nabla)$ is a connection on S. There is a K-linear map which we also denote by ∇ from $\Omega_S^p \otimes \mathcal{S} \to \Omega_S^{p+1} \otimes \mathcal{S}$ characterized by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \otimes \nabla(s)$$

for ω a local section of Ω_S^1 and s a local section of $\mathcal{S}$. We say that $(\mathcal{S}, \nabla)$ is integrable if the map $\nabla \circ \nabla : \mathcal{S} \to \Omega_S^2 \otimes \mathcal{S}$ is zero. In this case

$$\mathcal{S} \xrightarrow{\nabla} \Omega_S^1 \otimes \mathcal{S} \xrightarrow{\nabla} \Omega_S^2 \otimes \mathcal{S} \xrightarrow{\nabla} \cdots$$

is a complex. We let $H^i(\mathcal{S}, \nabla)$ denote the i-th hypercohomology group of this complex. When K is a field of characteristic zero, integrability also implies that $\mathcal{S}$ is locally free.

If (H, ∇_H) and (G, ∇_G) are two connections on S then there are natural connections $\nabla_H \otimes \nabla_G$ on $H \otimes G$ and $\nabla_{H,G}$ on $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$ characterized by the formulas

$$\nabla_H \otimes \nabla_G(h \otimes g) = \nabla_H(h) \otimes g + h \otimes \nabla_G(g)$$
$$\nabla_{H,G}(r)(h) = \nabla_G(r(h)) - r(\nabla_H(h))$$

for local sections h and g of H and G and a local section r of $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$. We let $\check{H} = \mathrm{Hom}(H, \mathcal{O}_S)$ and $\check{\nabla}_H = \nabla_{H, \mathcal{O}_S}$, which is a connection on $\check{H}$. It is easy to see that $\nabla_G \otimes \check{\nabla}_H$ equals $\nabla_{H,G}$ under the natural identification of $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$ with $G \otimes \check{H}$. We will need the following, easy to check, lemma.

LEMMA 1.0.1. *Suppose* $r \in \mathrm{Hom}_{\mathcal{O}_S}(H, \Omega_S^p \otimes G) \cong \Omega_S^p \otimes \mathrm{Hom}_{\mathcal{O}_S}(H, G)$. *Then*

$$\nabla_{H,G}(r)(s) = \nabla_G(r(s)) + (-1)^p r(\nabla_H(s)) .$$

Since we will use it frequently in the following we will record here the Čech definition of hypercohomology. (See also [H-1, Chapter 1 §3].) Suppose $(\mathcal{S}^\bullet, d)$ is a bounded below complex of Abelian sheaves on a topological space S. Then we define the hypercohomology of $\mathcal{S}$ as follows: First let $\mathcal{U}$ be an ordered open cover of S. We have the Čech complexes

$$C^i(\mathcal{U}, \mathcal{S}^j) = \oplus\, \mathcal{S}^j(U)$$

where the sum runs over all intersections U of $i+1$ distinct elements of $\mathscr{U}$. Let $\check{\delta}: C^i(\mathscr{U}, \mathscr{S}^j) \to C^{i+1}(\mathscr{U}, \mathscr{S}^j)$ be the Čech co-boundary. We also have boundaries $d: C^i(\mathscr{U}, \mathscr{S}^j) \to C^i(\mathscr{U}, \mathscr{S}^{j+1})$.

Now let

$$C^n(\mathscr{U}, \mathscr{S}^{\cdot}) = \bigoplus C^p(\mathscr{U}, \mathscr{S}^q)$$

where the sum runs over $p+q=n$. For $c \in C^n(\mathscr{U}, \mathscr{S}^{\cdot})$, we let $c^{p,q}$ denote its p, q-th component. The hyper-coboundary

$$\partial: C^n(\mathscr{U}, \mathscr{S}^{\cdot}) \to C^{n+1}(\mathscr{U}, \mathscr{S}^{\cdot})$$

is defined as follows: For $c \in C^n(\mathscr{U}, \mathscr{S}^{\cdot})$, we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1}\check{\delta} c^{p,q-1} .$$

Then the hypercohomology of $\mathscr{S}$ with respect to $\mathscr{C}$, $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}, \mathscr{C})$, is defined to be $\mathrm{Ker}(\partial)/\mathrm{Image}(\partial)$ and $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot})$ is defined to be an appropriate limit of these groups over all ordered covers. In particular, if S is a scheme, $\mathscr{S}^{\cdot}$ is a complex of coherent sheaves and $\mathscr{C}$ is an affine open cover, then $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot})$ is naturally isomorphic to $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}, \mathscr{C})$. If in addition S is affine $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}) \cong H^{\cdot}(\Gamma(\mathscr{S}^{\cdot}))$.

1. Extensions of connections

Let S be smooth connected scheme over a field K of characteristic zero. Suppose (H, ∇_H) and (G, ∇_G) are integrable connections on S. The set of isomorphism classes of integrable extensions of (H, ∇_H) by (G, ∇_G) forms a group under Baer sum which we will call $\mathrm{Ext}(H, G)$.

PROPOSITION 1.1.1. $\mathrm{Ext}(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$.

Proof. Since ∇_H is integrable, H is locally free. Let $\mathscr{C}$ be an ordered affine open cover of S such that $H(U)$ is a free $\mathscr{O}_S(U)$-module for each $U \in \mathscr{C}$. Suppose we have an extension

$$0 \to (G, \nabla_G) \to (E, \nabla) \to (H, \nabla_H) \to 0$$

of connections. Let $U \in \mathscr{C}$. Since $H(U)$ is free, there exists an $\mathscr{O}_S(U)$-module section $s_U: H(U) \to E(U)$. Now let $h_U = \nabla \circ s_U - s_U \circ \nabla_H$. We claim that h_U is an $\mathscr{O}_S(U)$-module homomorphism from $H(U)$ into $\Omega^1_S \otimes G(U)$, i.e. an element of $\mathrm{Hom}_{\mathscr{O}_S}(H, \Omega^1_S \otimes G)(U)$. Indeed, for $f \in \mathscr{O}_S(U)$ and $v \in H(U)$,

$$h_U(fv) = \nabla(s_U(fv)) - s_U(\nabla_H(fv)) = \nabla(fs_U(v)) - s_U(df \otimes v + f\nabla_H v)$$
$$= df \otimes s_U(v) - f\nabla(s_U(v)) - (df \otimes s_U(v) + fs_U(\nabla_H v)) = fh_U(v) .$$

Let $s_{U,V} = s_U - s_V \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)(U \cap V)$. We claim that $(\{h_U\}, \{s_{U,V}\})$ is a hyper one-cocycle for the complex $(\Omega^{\cdot}_S \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$. First it is clear that $\{s_{U,V}\}$ is a one-cocycle for the sheaf $\mathrm{Hom}_{\mathscr{O}_S}(H, G)$. Second

$$\nabla_G \circ s_{U,V} - s_{U,V} \circ \nabla_H = \nabla \circ (s_U - s_V) - (s_U - s_V) \circ \nabla_H = h_U - h_V .$$

Finally, since

$$\nabla \circ \nabla \circ s_U = \nabla \circ s_U \circ \nabla_H + \nabla \circ h_U = h_U \circ \nabla_H + \nabla_G \circ h_U = \nabla_{H,G}(h_U) ,$$

(using Lemma 1.0.1) ∇ is integrable iff $\nabla_{H,G}(h) = 0$.

Moreover, suppose $\{s'_U\}$ is another collection of sections

$$s'_U : H(U) \to E(U), \; h'_U = \nabla' \circ s'_U - s'_U \circ \nabla$$

and $s'_{U,V} = s'_U - s'_V$. Then $r_U = s'_U - s_U \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)$ and

$$h'_U = h + \nabla \circ r_U - r_U \circ \nabla_H = h + \nabla_G \circ r_U - r_U \circ \nabla_H = h + \nabla_{H,G}(r_U) .$$

And so $(\{h_U\}, \{s_{U,V}\}) - (\{h'_U\}, \{s'_{U,V}\})$ is the hyper-boundary of $\{r_U\}$. Thus we get a natural map from

$$\mathrm{Ext}(H, \mathscr{O}_X) \quad \text{into} \quad H^1(\mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) .$$

It is easy to see that this map is a homomorphism.

We can make a map back as follows. Given a hyper-cocycle $(\{h_U\}, \{s_{U,V}\})$ for the complex $(\Omega^{\cdot}_S \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$, let E be the sheaf determined by the condition that $E(U) = G(U) \oplus H(U)$ with gluing data

$$(w, v) \to (w + s_{U,V}, v)$$

on $U \cap V$. We then put a connection ∇ on E by setting

$$\nabla(w, v) = (\nabla_G w + h_U(v), \nabla_H v)$$

for local sections w and v of G and H on U. One can check easily that E is an extension of H by G and that this construction gives the inverse to the map above. □

COROLLARY 1.1.2. *$\mathrm{Ext}(H, \mathscr{O}_S)$ is a K vector space and hence is uniquely divisible.*

COROLLARY 1.1.3. *Suppose S is affine and S' is a non-empty affine open of S. Then $\mathrm{Ext}(H, \mathscr{O}_S)$ injects into $\mathrm{Ext}(H \otimes \mathscr{O}_{S'}, \mathscr{O}_{S'})$.*

We note that taking duals yields an isomorphism between $\mathrm{Ext}(G, H)$ and $\mathrm{Ext}(\check{H}, \check{G})$. Also, upon identifying $(\check{G})^\vee$ with G, $\check{\nabla}_{\check{G}} = \nabla_G$.

LEMMA 1.1.4. *The diagram*

$$\begin{array}{ccc} \mathrm{Ext}(H, G) & \rightarrow & H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\check{G}, \check{H}) & \rightarrow & H^1(\check{H} \otimes G, \check{\nabla}_H \otimes \nabla_G) \end{array}$$

anti-commutes, where the horizontal arrows are the isomorphisms given by the proposition and the right vertical arrow is the evident one.

Proof. Since the assertion is local, we may suppose H and G are free. Suppose (E, ∇) is an extension of H by G and $s: H \to E$ is a section. Then $h = \nabla \circ s - s \circ \nabla_H$ is an element of $\mathrm{Hom}_{\mathcal{O}_S}(H, \Omega_S^1 \otimes G)$ which represents the image of the isomorphism class of E in

$$H^1(\mathrm{Hom}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) .$$

The image k of h in $\mathrm{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ is determined by

$$k(w)(v) = w(h(v)) = w((\nabla \circ s - s \circ \nabla_H)(v))$$

where v is a section of H and w is a section of $\check{G}$.

Now $(\check{E}, \check{\nabla})$ is an extension of $\check{G}$ by $\check{H}$ and the homomorphism t determined by

$$t(w)(e) = w(e - s \circ \pi(e))$$

is a section, where $\pi: E \to H$ is the projection, e is a section of E and w is a section of $\check{G}$. Hence, $g = \check{\nabla} \circ t - t \circ \nabla_{\check{G}}$ is an element of $\mathrm{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ which represents the image of the isomorphism class of $\check{E}$ in

$$H^1(\mathrm{Hom}(\check{G}, \check{H}), \nabla_{\check{G},\check{H}}) .$$

Now

$$g(w)(v) = (\check{\nabla} \circ t - t \circ \nabla_{\check{G}})(w)(e)$$

where $e = s(v)$ and

$$\begin{aligned} \check{\nabla} \circ t(w)(e) &= d(w(e - s(\pi(e)) - w(\nabla(e) - s(\pi(\nabla(e)))) \\ &= - w(\nabla \circ s(v) - s \circ \nabla_H(v)) = - k(w)(v) \end{aligned}$$

since $\pi(s(v)) = v$ and $\pi\nabla(e) = \nabla_H(\pi(e))$. The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e)) = 0 . \quad \square$$

Suppose W is an $\mathcal{O}_S$ submodule of H. We let [W] denote the smallest subconnection of H containing W.

2. The Gauss-Manin connection

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If $\mathcal{S}^\cdot$ is a complex, $\mathcal{S}^\cdot(k)$ will denote the complex obtained from $\mathcal{S}^\cdot$ by setting $\mathcal{S}^i(k) = \mathcal{S}^{i+k}$. For any scheme Y over K will let $K[Y]$ denote $\Gamma(\mathcal{O}_Y)$.

Suppose S is a smooth connected affine scheme over K. Suppose $f: X \to S$ is a smooth morphism, Z is a closed subscheme of X, smooth over S. Suppose T is either $\mathrm{Spec}(K)$ or S. Then we define the subcomplex $\Omega^\cdot_{X/T,Z}$ of $\Omega^\cdot_{X/T}$ by the exactness of the sequence.

$$0 \to \Omega^\cdot_{X/T,Z} \to \Omega^\cdot_{X/T} \to \Omega^\cdot_{Z/T} \to 0 .$$

When $T = \mathrm{Spec}(K)$ we drop it from the notation. It follows that $\Omega^i_{X/S,Z} = \Omega^i_{X/S}$ for $i > \dim_S Z$. Note that $\Omega^0_{X,Z} = \Omega^0_{X/S,Z}$ is the sheaf of ideals of Z on X. We define $H^i_{DR}(X/S, Z)$ to be the i-th hypercohomology group of the complex $\Omega^\cdot_{X/S,Z}$. We set $H^i_{DR}(X/S) = H^i_{DR}(X/S, \emptyset)$. If X is affine, then $H^i_{DR}(X/S, Z)$ is the i-th cohomology group of the complex of $K[S]$ modules $\Gamma(\Omega^\cdot_{X/S,Z})$. If X is affine, K has characteristic zero and U is a dense open subscheme of X then the natural map from $H^i_{DR}(X/S, Z)$ to $H^i_{DR}(U/S, U \cap Z)$ is an injection.

From the last short exact sequence with $T = S$, we obtain a long exact sequence

$$\text{(2.1)} \qquad \ldots \to H^{i-1}_{DR}(Z/S) \to H^i_{DR}(X/S, Z) \to H^i_{DR}(X/S) \to \ldots$$

The Gauss-Manin connection $\nabla: H^i_{DR}(X/S, Z) \to \Omega^1_S \otimes H^i_{DR}(X/S, Z)$ is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

$$\text{(2.2)} \quad 0 \to f^*\Omega^1_S \otimes \Omega^\cdot_{X/S,Z}(-1) \to \Omega^\cdot_{X/S,Z}/f^*\Omega^2_S \otimes \Omega^\cdot_X(-2) \to \Omega^\cdot_{X/S,Z} \to 0$$

(which is exact because X and Z are smooth over S). It is an integrable connection. If K has characteristic zero and f is surjective and has geometrically connected fibers, then $H^0_{DR}(X/S) = K[S]$ and the Gauss-Manin connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that S is an affine curve over K and $Z = \varnothing$. Then the short exact sequence (2.2) becomes

$$0 \to f^*\Omega^1_S \otimes \Omega^{\cdot}_{X/S}(-1) \to \Omega^{\cdot}_{X/S} \to \Omega^{\cdot}_{X/S} \to 0\ .$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$\text{(2.3)} \qquad \ldots \to H^i_{DR}(X/S) \xrightarrow{\nabla} \Omega^1_S \otimes H^i_{DR}(X/S) \to H^{i+1}_{DR}(X) \to H^{i+1}_{DR}(X/S) \xrightarrow{\nabla} \ldots$$

3. Sections of a family and extensions of connections

Suppose now S is a smooth connected affine curve over a field K of characteristic zero and $f: X \to S$ is a smooth proper morphism of schemes over K, with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose Z is a closed subscheme of X finite over S. Suppose the normalization $n: \tilde{Z} \to Z$ of Z is smooth over S. After repeated blowing ups at closed points we find a scheme $m: \tilde{X}' \to X$, which contains $\tilde{Z}$ and is such that the restriction of m to $\tilde{Z}$ is n. Let $\tilde{X}$ equal the complement in $\tilde{X}'$ of the singular locus of $\tilde{X}'/S$. This locus is a closed subscheme of $\tilde{X}'$ disjoint from $\tilde{Z}$. The long exact sequence 2.1 becomes

$$\text{(3.1)} \qquad 0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(\tilde{Z}/S, \tilde{Z}) \to H^1_{DR}(\tilde{X}/S) \to 0$$

Let H denote the pullback of $H^1_{DR}(\tilde{X}/S, \tilde{Z})$ by means of the horizontal monomorphism from $H^1_{DR}(X/S)$ into $H^1_{DR}(\tilde{X}/S)$. We claim that H is independent of the choice of $\tilde{X}$. Indeed, there exists a non-empty affine open subscheme S' of S such that the map from $\tilde{X} \times_S S'$ to $X' = : X \times_S S'$ is an isomorphism. If $Z' = Z \times_S S'$, then Z' is smooth over S' and it is easy to see that $H \otimes K[S'] \cong H^1_{DR}(X'/S', Z')$. Hence H is an extension of the connection $H^1_{DR}(X'/S', Z')$ on S' to a connection on S. Since such an extension is unique if it exists, it follows that H is independent of the choice of $\tilde{X}$ and so we set $H^1_{DR}(X/S, Z) = H$. We obtain from the previous exact sequence, a natural exact sequence

$$0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0\ .$$

For a section s of X/S, we will also use s to denote the induced reduced closed subscheme $s(S)$ of X when convenient. Now suppose s and t are two distinct sections of X/S. Let $Z = s \cup t$. Then $\tilde{Z}$, the normalization of Z, is just two disjoint copies of S and so is étale over S. (The sections s and t

induce maps from S to $\tilde{Z}$ which we denote by the same names.) The map $t^* - s^*: K[\tilde{Z}] \to K[S]$ is horizontal, surjective and its kernel is the image of $K[S]$ under the map in (3.1). Hence we obtain a horizontal exact sequence

$$0 \to K[S] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0$$

and so an extension of $H^1_{DR}(X/S)$ by the trivial connection. We let $E(s, t)$ denote this extension if $s \neq t$ and $E(s, s)$ denote the trivial extension of $H^1_{DR}(X/S)$ by $K[S]$. We call the class of $E(s, t)$ in $\mathrm{Ext}(H^1_{DR}(X/S), K[S])$ $M(s, t)$.

PROPOSITION 1.3.1. *Suppose r, s, t are sections of X/S. Then*

$$M(r, t) = M(r, s) + M(s, t) .$$

In particular, $M(r, s) = -M(s, r)$.

Proof. If r, s and t are not distinct the proposition is obvious from the definitions. Therefore suppose that r, s and t are distinct. If T is a subset of $\{r, s, t\}$ let $Z_T = \bigcup_{u \in T} u$. Either by replacing X by $\tilde{X}$ or by shrinking S and using Corollary 1.1.3 we may assume that $Z_{\{r,s,t\}}$ is étale over S. Let $\mathscr{F}_T$ denote the complex $\Omega^{\bullet}_{X/S, Z_T}$. We set $H(T) = H^1_{DR}(X/S, Z_T)$. Then from the exact sequence of complexes

$$0 \to \mathscr{F}_{\{r,s,t\}} \to \mathscr{F}_{\{r,s\}} \otimes \mathscr{F}_{\{s,t\}} \to \mathscr{F}_{\{s\}} \to 0$$

(where the first map is the diagonal and the last is the difference) we obtain an exact sequence

$$0 \to H(r, s, t) \to H(r, s) \oplus H(s, t) \to H(s)$$

moreover, $H(s) \cong H^1_{DR}(X/S)$ and the last map is the difference of the maps from $H(r, s)$ and from $H(s, t)$ to $H^1_{DR}(X/S)$ (and is, in particular, a surjection).

Next from the exact sequence of complexes

$$0 \to \mathscr{F}_{\{r,s,t\}} \to \mathscr{F}_{\{r,t\}} \to \mathscr{S} \to 0$$

where $\mathscr{S}$ is the complex $(\mathscr{I}_{\{r,t\}}/\mathscr{I}_{\{r,s,t\}} \to 0 \to \ldots) \cong (K[S] \to 0 \to \ldots)$ we obtain an exact sequence

$$0 \to K[S] \to H(r, s, t) \to H(r, t) \to 0$$

Moreover the first map is the composition of the map from $K[Z_{\{r,s,t\}}]$ into $H(r, s, t)$ and the map h from $K[S]$ into $K[Z_{\{r,s,t\}}]$ characterized by $r^*h(f) = t^*h(f) = 0$ and $t^*h(f) = f$. It follows from this that $H(r, t)$ is the

Baer sum of $H(r,s)$ and $H(s,t)$. Since all the maps discussed above are horizontal this statement is true on the level of connections as well. This proves the proposition. □

Suppose X' is a smooth scheme over S and $g: X' \to X$ is an S-morphism. Then the natural map $g^*: H^1_{DR}(X/S) \to H^1_{DR}(X/S)$ induces a natural map $g^*: \mathrm{Ext}(H^1_{DR}(X'/S), K[S]) \to \mathrm{Ext}(H^1_{DR}(X/S), K[S])$. By the naturality of all our constructions we have:

PROPOSITION 1.3.2. *Suppose X'/S has geometrically connected fibers and s and t are two sections of X'/S. Then*

$$M(g \circ s, g \circ t) = g^* M(s,t) .$$

Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Then

$$(\Omega^{\cdot}_{X/S}, d_{X/S}) \cong K[S] \otimes (\Omega^{\cdot}_{X_0/K}, d_{X_0/K})$$

and so in particular,

$$H^1_{DR}(X/S) \cong K[S] \otimes H^1_{DR}(X_0/K)$$

and the Gauss-Manin connection

$$\nabla: H^1_{DR}(X/S) \to \Omega^1_S \otimes_{K[S]} H^1_{DR}(X/S)$$

is (d, id). If $H = H^1_{DR}(X/S)$, it follows from this that

$$\mathrm{Ext}(H, K[S]) \cong H^1(\check{H}, \check{\nabla}) \cong \mathrm{Hom}_K(H^1_{DR}(X_0/K), H^1_{DR}(S/K)) .$$

Explicitly, this last isomorphism can be described as follows:

$$\text{if} \quad h \in \mathrm{Hom}(H, \Omega^1_S) \cong \Omega^1_S \otimes \check{H} ,$$

then $h \bmod \check{\nabla}\check{H}$ goes to the map $(\omega \in H^1_{DR}(X_0/K) \to h(1 \otimes \omega) \bmod dK[S])$.

PROPOSITION 1.3.3. *Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Suppose u and v are two morphisms from S to X_0 and $s = (id, u)$ and $t = (id, v)$. Then $M(s,t)$ is $v^* - u^*$ as an element of* $\mathrm{Hom}_K(H^1_{DR}(X_0/K), H^1_{DR}(S/K))$.

Proof. We may suppose that $s \cap t = \varnothing$. Let $Z = s \cup t$. Suppose $h: H^1_{DR}(X/S) \to H^1_{DR}(X/S, Z)$ is a section. Let $(\{\omega_U\}, \{f_{U,V}\})$ be a one-hypercocycle for $(\Omega^{\cdot}_{X_0/K}, d_{X_0/K})$ and $[\omega]$ the image the class of $1 \otimes (\{\omega_U\}, \{f_{U,V}\})$ in $H^1_{DR}(X/S)$. Then $\nabla[\omega] = 0$. We wish to compute $\nabla h([\omega]) - h(\nabla[\omega]) = \nabla h([\omega])$. We will abuse notation and identify ω_U with $1 \otimes \omega_U$ in $\Omega^1_X(U)$ and $f_{U,V}$ with $1 \otimes f_{U,V}$ in $\mathscr{O}_X(U \cap V)$. Let $\bar{\omega}_U$ denote the image of ω_U in $\Omega^1_{X/S}(U)$. Then $h([\omega])$ is the class of

$$(\{\bar{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain $\{g_U\}$ with coefficients in $\mathscr{O}_X$ such that

$$s^*f_{U,V} = u^*f_{U,V} = s^*(g_U - g_V) \quad \text{and} \quad t^*f_{U,V} = v^*f_{U,V} = t^*(g_U - g_V) .$$

Let $\eta_U = \omega_U - dg_U$. Now

$$s^*\eta_U - s^*\eta_V = s^*df_{U,V} - s^*d(g_U - g_V) = 0$$

by the conditions that $\{g_U\}$ must satisfy and the fact that $(\{\omega_U\}, \{f_{U,V}\})$ is a hypercocycle. Similarly, $t^*\eta_U - t^*\eta_V = 0$. Let η_s and η_t be the the elements of Ω_S^1 determined by the cocycles $\{s^*\eta_U\}$ and $\{t^*\eta_U\}$ respectively.

Now to compute $\nabla h([\omega])$ we must lift $\bar{\omega}_U - d_{X/S}g_U$ to a section of $\Omega^1_{X,Z}$. Let $e_{s,U}$ and $e_{t,U}$ be elements of $\mathscr{O}_X(U)$ such that $s^*e_{s,U} = 1$ $t^*e_{t,U} = 0$, $t^*e_{t,U} = 1$ and $s^*e_{t,U} = 0$. These elements exist since Z is étale over S. Then $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$ is such a lifting. To compute $\nabla h([\omega])$ we must take the hyper-coboundary of $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$. It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0) .$$

The class of this hypercocycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1 \quad \text{in} \quad \Omega_S^1 \otimes H^1_{DR}(X/S, Z)$$

(recall that we've determined a map of $K[S]$ into $H^1_{DR}(X/S, Z)$). Hence $\nabla h([\omega]) = \eta_t - \eta_s$.

The proposition now follows from the fact that

$$(\{\eta_s + ds^*g_U\}, \{s^*g_U - s^*g_V\}) = u^*(\{\omega_U\}, \{f_{U,V}\})$$

and

$$(\{\eta_t + dt^*g_U\}, \{t^*g_U - t^*g_V\}) = v^*(\{\omega_U\}, \{f_{U,V}\}) . \quad \square$$

COROLLARY 1.3.4. *If, in the above, u and v are constant, then* $M(s, t) = 0$.

4. ABELIAN SCHEMES

Suppose now that A is an Abelian scheme over S. Let $m: A \times_S A \to A$ be the addition law and e the zero section. For $s, t \in A(S)$, let $M(s) = M(e, s)$ and $s + t = m(s, t)$.

THEOREM 1.4.1. *The map M from $A(S)$ to* $\mathrm{Ext}(H^1_{DR}(A/S), K[S])$ *is a homomorphism.*

Proof. Let s and t be elements of $A(S)$. Define the map $g: A \to A$ by $g = m \circ (id, t \circ f)\,(g(x) = x + t(f(x)))$. Then $g^*: H^1_{DR}(A/S) \to H^1_{DR}(A/S)$ is the identity so that $g^*M(e, s) = M(e, s)$ on the one hand and $g^*M(e, s) = M(t, s + t)$ by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$

by Proposition 1.3.1. □

Let (B, τ) denote the $K(S)/K$ trace of $A_{K(S)}$ (see [L-AV]). In particular, B is an Abelian scheme over K and $\tau: B \times \operatorname{spec}(K(S)) \to A_{K(S)}$ is a homomorphism. Since K has characteristic zero τ is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of $A_{K(S)}$ defined over over K. The morphism τ extends uniquely to an S-morphism $\bar{\tau}: B \times_K S \to A$. It follows that $B(K)$ maps naturally into $A(S)$. We call the elements s of $A(S)$ such that ns is in the image of $B(K)$, the constant sections of A/S.

PROPOSITION 1.4.2. *The kernel of* M *contains all constant sections of* A/S.

Proof. Let s be a constant section of A/S. Then there exists a positive integer n such that $ns = \bar{\tau} \circ (t \times id)$ where $t \in B(K)$. Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^* M(t \times id) = 0$. Since

$$\operatorname{Ext}(H^1_{DR}(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows. □

We wish to prove the conserve of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. *The kernel of* M *is precisely the group of all constant sections of* A/S.

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

5. THE ALGEBRAIC PROOF

a. *Differentials with logarithmic singularities*

(See [K] § 1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and cross normally relative to T. Let $W = X - Z$ and $\tilde{Z}$ the disjoint union of the irreducible components of Z. Let $(\Omega^{\cdot}_{X/T}(\mathrm{Log}(Z)), d)$ denote the complex of differentials on X/T with logarithmic singularities along Z. (When $T = K$, we drop T from the notation.) When T has characteristic zero, which we will now assume, the i-th hypercohomology group of this complex is naturally isomorphic to $H^i_{DR}(W/T)$. We have a natural short exact sequence of complexes

$$0 \to \Omega^{\cdot}_{X/T} \to \Omega^{\cdot}_{X/T}(\mathrm{Log}(Z)) \overset{\mathrm{Res}}{\to} \Omega^{\cdot}_{\tilde{Z}/T}(-1) \to 0$$

From which, upon taking cohomology, we obtain the long exact sequence:

$$(5.1) \qquad 0 \to H^1_{DR}(X/T) \to H^1_{DR}(W/T) \to H^0_{DR}(\tilde{Z}/T) \to H^2_{DR}(X/T) \to H^2_{DR}(W/T) \to H^2_{DR}(\tilde{Z}/T)$$

In addition, we have a short exact sequence of complexes

$$0 \to \Omega^1_S \otimes \Omega^{\cdot}_{X/S}(\mathrm{Log}(Z))(-1) \to \Omega^{\cdot}_X(\mathrm{Log}(Z)) \to \Omega^{\cdot}_{X/S}(\mathrm{Log}(Z)) \to 0\,.$$

The boundary maps in the long exact sequence of hypercohomology obtained from this short exact sequence are the Gauss-Manin connections $\nabla: H^i_{DR}(W/S) \to \Omega^1_S \otimes H^i_{DR}(W/S)$. Moreover the long exact sequence (5.1) is horizontal with respect to all the Gauss-Manin connections.

If D is any divisor on X, let $\eta_T(D)$ denote the cohomology class of D in $H^2_{DR}(X/T)$. Recall ([H-DR; 7.7]), if $\mathscr{L}$ is an ordered affine open cover of C and $\{f_U\}$ is a Čech one-cochain with with coefficients in $\mathscr{O}_A$ with respect to $\mathscr{L}$ such that the divisor of f_U is the restriction of D to U, then $\eta_T(D)$ is the cohomology class represented by the hyper one-cocycle $(0, \{d_{C/T}\mathrm{Log}(f_{U,V})\}, 0)$, where $f_{U,V} = f_U/f_V (U < V)$. Suppose now that T is affine. Then $H^0_{DR}(\tilde{Z}/T)$ is naturally isomorphic to the group of divisors on X supported on Z with coefficients in $K[T]$.

LEMMA 1.5.1. *Suppose D is a divisor on X supported on Z, then the image of D in $H^2_{DR}(X/T)$ via the appropriate map in (5.1) is equal to $\eta_T(D)$.*

Proof. This is essentially Proposition 7.6 of [H]. We carry out the proof in order to "straighten out" the sign.

Let $\mathscr{L}$ be an affine open cover of X and $\{f_U\}$ is a Čech one-chain with coefficients in $\mathscr{O}_X$ with respect to $\mathscr{L}$ such that the divisor of f_U is the restriction of D to U. Then, $d\mathrm{Log}(f_U) \in \Omega^1_{X/T}(\mathrm{Log}(Z))(U)$ and $\mathrm{Res}(d\mathrm{Log}(f_U))$ is the image of the image of D in $H^0_{DR}(\tilde{Z}/T) \cong \mathscr{O}_{\tilde{Z}}(U)$. It follows that the image of D in $H^2_{DR}(X/T)$ is the class of the hyper-coboundary of $(\{d\mathrm{Log}(f_U)\},0)$ which is $\eta_T(D)$ by definition. □

By a properly semi-stable curve over S, we mean a curve over S such that the irreducible components of the closed fibers are smooth and cross normally. (The irreducible components do not have to be smooth if the curve is only semi-stable.)

COROLLARY 1.5.2. *Suppose R is a smooth connected curve over a field K and X is a properly semi-stable curve over R smooth over K. Suppose U is a non-empty open subset of R and $Y = R - U$. Then the kernel of the natural map from $H^2_{DR}(X)$ into $H^2_{DR}(X_U)$ is generated by $\{\eta(D)\}$ where D runs over the irreducible components of X_Y.*

Proof. This follows from the lemma and the exact sequence (5.1), since the closed fibers of C/T are unions of smooth hypersurfaces of C which cross normally. □

LEMMA 1.5.3. *With notation as in the above corollary, if R is affine and X is smooth over R then the map from $H^2_{DR}(X) \to H^2_{DR}(X_U)$ is an injection.*

Proof. For a closed point x of R, let X_x denote the fiber above x. Since all the fibers of X over S are smooth, it follow from the corollary that the the kernel of the map $H^2_{DR}(X) \to H^2_{DR}(X_U)$ is generated by $\{\eta(X_x)\}$ where x runs over the closed points of Y. Now $\eta(X_x)$ is the pull-back of $\eta(x) \in H^2_{DR}(R)$. As this latter group is zero, this proves the lemma. □

b. *End of algebraic proof*

First by using the functoriality of M, Proposition 1.3.2, and the fact that every Abelian variety over S is the quotient of a Jacobian over S we may assume that A is the Jacobian of a smooth proper curve C over S. By Proposition 1.1.1 and the long exact sequence (2.3), $\mathrm{Ext}(H^1_{DR}(C/S)^\vee, K[S])$ maps naturally into $H^2_{DR}(C)$. Moreover, since C is a proper smooth connected curve over $S, H^1_{DR}(C/S)$ is canonically isomorphic to $H^1_{DR}(C/S)^\vee$. The fact we need to finish the proof is:

PROPOSITION 1.5.4. *Let s and t be two elements of $C(S)$. The class $\eta(t - s)$ is equal to the image of $M(s, t)$ in $H^2_{DR}(C)$.*

By the previous lemma and the functoriality of η we may shrink S to suppose that $s \cap t = \varnothing$. To prove the proposition, we need the next lemma.

Suppose now that $T = S, Z = s \cup t$ and $X = C$. Then the exact sequence (5.1) becomes:

$$(5.2) \qquad 0 \to H^1_{DR}(C/S) \to H^1_{DR}(W/S) \to H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S) \to 0$$

Furthermore $H^2_{DR}(C/S)$ is canonically isomorphic to $K[S]$ with generator $\eta_S(s) = \eta_S(t)$ and so the kernel of $H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S)$ is a principal $K[S]$ module with generator $D = s - t$. Using this generator, (5.2) yields an extension $B_{s,t}$ of the connection $(K[S], d)$ by $(H^1_{DR}(C/S), \nabla)$.

LEMMA 1.5.5. *Identifying* $H^1_{DR}(C/S)$ *with* $H^1_{DR}(C/S)^\vee$, *the extension* $B_{s,t}$ *is isomorphic to the dual of* $E_{s,t}$.

Proof. Regarding the complexes $\Omega^\bullet_{C/S,Z}$ and $\Omega^\bullet_{C/S}(\mathrm{Log}(Z))$ as subcomplexes of $\Omega^\bullet_{W/S,Z}$ the wedge product gives a product from

$$\Omega^\bullet_{C/S.Z} \times \Omega^\bullet_{C/S}(\mathrm{Log}(Z))$$

into $\Omega^\bullet_{C/S}$ which induces a pairing

$$(\ ,\ ): H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S) \to H^2_{DR}(C/S) \cong K[S]\ .$$

This pairing is compatible with the exact sequences

$$0 \to H^0(C, \Omega^1_{C/S}) \to H^1_{DR}(C/S, Z) \to H^1(C, \Omega^0_{X/S}) \to 0$$

$$0 \to H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)) \to H^1_{DR}(W/S) \to H^1(C, \mathscr{O}_C) \to 0$$

arising from the Hodge to de Rham spectral sequences for hypercohomology (which degenerate). In other words, the image of $H^0(C, \Omega^1_{C/S})$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z))$ in $H^1_{DR}(W/S)$ and if we identify $\Omega^0_{X,Z}$ with $\mathscr{O}_C(Z)$ and $\Omega^1_{C/S}(\mathrm{Log}(Z))$ with $\Omega^1_{C/S}(-Z)$ the pairings induced on $H^0(C, \Omega^1_{C/S}) \times H^1(C, \mathscr{O}_C)$ and on

$$H^1(C, \Omega^0_{X,Z}) \times H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)))$$

are the natural ones. Since these pairings are non-degenerate, it follows that the pairing on $H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S)$ is non-degenerate.

It is also clear that the image of $H^0_{DR}(Z/S) \cong K[Z]$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^1_{DR}(C/S)$ in $H^1_{DR}(W/S)$ and that the pairing induced on $H^1_{DR}(C/S) \times H^1_{DR}(C/S)$ is the natural one.

The lemma will follow from the following claim: Let ι denote the map from $K[Z]$ to $H^1_{DR}(C/S, Z)$ and Res the map from $H^1_{DR}(W/S)$ to $K[Z]$. Let $T_{Z/S}$ denote the trace from $K[Z]$ to $K[S]$. Suppose $c \in K[Z]$ and $\omega \in H^1_{DR}(W/S)$. Then

$$(\iota(c), w) = -T_{Z/S}(c\operatorname{Res}(\omega)).$$

Indeed, if $s^*c = 0, t^*c = 1, \operatorname{Res}_s(\omega) = 1$ and $\operatorname{Res}_t(\omega) = -1$ then $-T_{Z/S}(c\operatorname{Res}(\omega)) = 1.$

To prove this claim we may shrink S. Hence, we may assume first that $\{U, V\}(U < V)$ is an ordered affine open cover of C such that $U = C - s$ and $V = C - t$, second that $\iota(c)$ is represented by a hypercocycle of the form $\partial(\{g_U, g_V\})$ where $s^*g_U = s^*c$ and $t^*g_V = t^*c$ and third, since the composition $H^0(C, \Omega^1_{C/S}(\operatorname{Log}(Z)) \to H^1_{DR}(W/S) \to K[Z]$ is surjective, that w is in the image of $H^0(C, \Omega^1_{C/S}(\operatorname{Log}(Z)))$, i.e., w is represented by a hypercocycle of the form $(\{\omega_U, \omega_V\}, 0)$ where $\omega_U = \omega = \omega_V$ on $U \cap V$ for some $\omega \in H^0(C, \Omega^1_{C/S}(\operatorname{Log}(Z)))$. It follows that $(\iota(c), w)$ as an element of $H^1_{DR}(C, \Omega^1_{C/S}) \cong H^2_{DR}(C/S)$ is represented by the cocycle $\{\nu_{U,V}\}$ with $\nu_{U,V} = (g_V - g_U)\omega$. Since the image of this element in $K[S]$ is

$$\begin{aligned}\operatorname{Res}_s(-g_U\omega) + -\operatorname{Res}_t(g_V\omega) &= -(s^*g_U\operatorname{Res}_s(\omega) + t^*g_V\operatorname{Res}_t(\omega)) \\ &= -T_{T/S}(c\operatorname{Res}(\omega)).\end{aligned}$$

this establishes the claim and the lemma. □

End of proof of Proposition 1.5.4

Consider the commutative diagram of complexes of sheaves with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{C/S}(-1) & \to & \Omega^{\cdot}_C & \to & \Omega^{\cdot}_{C/S} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{C/S}(\operatorname{Log}(Z))(-1) & \to & \Omega^{\cdot}_C(\operatorname{Log}(C)) & \to & \Omega^{\cdot}_{C/S}(\operatorname{Log}(Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{Z/S}(-2) & \to & \Omega^{\cdot}_Z(-1) & \to & \Omega^{\cdot}_{Z/S}(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0\,. & &
\end{array}$$

If we take hyper-cohomology of this diagram we obtain a commutative diagram

$$\begin{array}{ccccccc} & & & & H^1_{DR}(C/S) & \overset{\nabla}{\to} & \Omega^1_S \otimes H^1_{DR}(C/S) \\ & & & & \downarrow & & \downarrow \\ & & & & H^1_{DR}(W/S) & \overset{\nabla}{\to} & \Omega^1_S \otimes H^1_{DR}(W/S) \\ & & & & \downarrow & & \\ & & H^0_{DR}(Z) & \to & H^0_{DR}(Z/S) & & \\ & & \downarrow & & \downarrow & & \\ H^1_{DR}(C/S) \overset{\nabla}{\to} \Omega^1_S \otimes H^1_{DR}(C/S) & \to & H^2_{DR}(C) & \to & H^2_{DR}(C/S) & & \end{array}$$

with exact rows and columns in which the bottom row is part of the Leray long exact sequence. Let a be the element in $H^0_{DR}(Z)$ corresponding to the divisor $s - t$. The image of a in $H^2_{DR}(C)$ is $\eta(s - t)$ by Lemma 1.5.1. On the other hand the image of a in $H^0_{DR}(Z/S)$ is our chosen generator of the kernel of the map to $H^2_{DR}(C/S)$. In particular, it is the image of an element b of $H^1_{DR}(W/S)$ and $\nabla(b)$ is the image of an element c of $\Omega^1_S \otimes H^1_{DR}(C/S)$ whose image in $H^2_{DR}(C/S)$ is the same as that of a by an elementary diagram chase. On the other hand, the image of c in $H^1(H^1_{DR}(C/S), \nabla)$ is the class corresponding to the extension $B_{s,t}$ by definition (see Proposition 1.1.1) which is, after identifying $H^1_{DR}(C/S)$ with $H^1_{DR}(C/S)^\vee$, $-M(s, t)$ by Lemma 1.5.1 and Lemma 1.5.3. Hence the image of $M(s, t)$ in in $H^2_{DR}(C)$ is $-\eta(s - t) = \eta(t - s)$ as required. □

Now we are in a position to prove the Theorem 1.4.3. We will suppose $M(s, t) = 0$ which amounts to $\eta_S(t - s) = 0$ by the Proposition 1.5.1. Recall, that A is the Jacobian of C/S. Let d denote the divisor class of $t - s$ in $A(K(C))$. We will show that the canonical height of d is zero. We may replace S by a finite étale cover and complete C to a properly semi-stable curve $\tilde{C}$ over the completion $\tilde{S}$ of S which is smooth over K. Let D be a **Q**-rational divisor on $\tilde{C}$ which is perpendicular (under the intersection pairing) to all the irreducible components of all the fibers of $\tilde{C}/\tilde{S}$ and whose restriction to C is $t - s$. Such a divisor exists by the function field analogue of Theorem 1.3 of [Hr] (see also Theorem 5.1 (i) of [Ch]). It follows that the image of $\eta(D)$ in $H^2_{DR}(C)$ is $\eta(t - s) = 0$. Corollary 1.5.2 implies that $\eta(D)$ is in the span of $\{\eta(Y)\}$ where Y runs over the irreducible component of the closed fibers above $\tilde{C} - C$. In particular, $D \cdot D = 0$ using Theorem 7.8.2 of [H]. On the other hand, $D \cdot D$ is -2 times the canonical height of d by the function field analogue of Theorem 5.1 of [Ch]. It now follows from Theorem 5.4.1 of [L], that the image of $t-s$ in $J(C)$ is a constant section which completes the proof. $\square$

6. The Analytic Proof

In this section we will suppose $K = \mathbf{C}$.

a. *The Poincaré Lemma*

Suppose $(\mathscr{S}, \nabla)$ is a sheaf on S^{an} with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathscr{S} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \Omega^2_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf $\mathscr{S}^{\nabla}$. Hence,

Proposition 1.6.1. *$H^i(\mathscr{S}, \nabla)$ is naturally isomorphic to $H^i(S, \mathscr{S}^{\nabla})$.*

Remark. As in Proposition 1.1.1, $H^1(\mathscr{S}, \nabla)$ is isomorphic to $\mathrm{Ext}(\mathscr{S}^{\vee}, \mathscr{O}_{S^{an}})$. We can describe the isomorphism from $H^1(\mathscr{S}, \nabla)$ to $H^1(S, \mathscr{S}^{\nabla})$ explicitly as follows: Let h be an element of $H^1(\mathscr{S}, \nabla)$. Let $\mathscr{C}$ is a covering of S by open disks. Suppose $\mathscr{E}$ is an extension of $\mathscr{S}^{\vee}$ by $\mathscr{O}_{S^{an}}$ corresponding to h. Then $\mathscr{E}^{\vee}$ is an extension of $\mathscr{O}_{S^{an}}$ by $\mathscr{S}$. For each $U \in \mathscr{C}$, there exists an $s_U \in \mathscr{E}^{\vee}(U)^{\nabla}$ which maps to 1 in $\mathscr{O}_{S^{an}}(U)$. Then the image h in $H^1(S, \mathscr{S}^{\nabla})$ is the class of the cocycle $\{(U, V) \to s_U - s_V\}$.

Suppose, X is a smooth proper S-scheme and Z is a subscheme of X which is either empty or finite over S. We will define the Betti homology sheaf $\mathscr{H}_1(X/S, Z, \mathbf{Z})$ on S^{an} as follows. If Z is smooth over S, we define $\mathscr{H}_i(X/S, Z, \mathbf{Z})$ to be the sheaf associated to the presheaf

$$U \to H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z}) ,$$

(this latter group is the Betti homology of $f^{-1}(U)$ relative to $f^{-1}(U) \cap Z$). More generally, let S' be a non-empty affine open subset of S such that $Z' = Z \times_S S'$ is étale over S'. Let $X' = X \times_S S'$ and let ι denote the inclusion morphisms $X' \to X$, $Z' \to Z$ and $S' \to S$. We set

$$\mathscr{H}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathscr{H}_i(X'/S', Z', \mathbf{Z}) .$$

This is independent of the choice of S'. We also set

$$\mathscr{H}_i(X/S, \mathbf{Z}) = \mathscr{H}_i(X/S, \varnothing, \mathbf{Z}) \text{ and } \mathscr{H}_1(X/S, Z, \mathbf{C}) = \mathscr{H}_1(X/S, Z, \mathbf{Z}) \otimes \underline{\mathbf{C}} .$$

Suppose s and t are two distinct sections of X/S and $Z = s \cup t$. Suppose S' is an affine open of S such that Z' is étale ove S' in the notation of the previous paragraph. We have exact sequences

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{H}_1(X/S, Z, \mathbf{Z}) \to \iota_* \mathscr{H}_0(Z'/S', \mathbf{Z}) \to \mathscr{H}_0(X/S, \mathbf{Z}) \to 0 .$$

and

$$0 \to \mathscr{H}_0(S'/S', \mathbf{Z}) \to \mathscr{H}_0(Z'/S', \mathbf{Z}) \to \mathscr{H}_0(X'/S', \mathbf{Z}) ,$$

where the first map is $t_* - s_*$. From which we derive the short exact sequence

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{H}_1(X/S, Z, \mathbf{Z}) \to \underline{\mathbf{Z}} \to 0 .$$

since $\iota_* \underline{\mathbf{Z}}|_{S'^{an}} \cong \underline{\mathbf{Z}}$. In particular, if U is an open disk in S^{an}, we have an exact sequence

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z})(U) \to \mathscr{H}_1(X/S, Z, \mathbf{Z})(U) \to \mathbf{Z} \to 0$$

We define the Betti cohomology sheaf $\mathscr{H}^1(X/S, Z, \mathbf{C})$ in the same way and it is easy to see that $\mathscr{H}^1(X/S, Z, \mathbf{C}) \cong \mathrm{Hom}(\mathscr{H}_1(X/S, Z, \mathbf{C}), \underline{\mathbf{C}})$. Also, it is known that if Z is étale over S then $\mathscr{H}^1(X/S, Z, \mathbf{C}) \cong R^1_{f_*} \mathscr{S}_Z$ where $\mathscr{S}_Z$ is the subsheaf of $\underline{\mathbf{C}}$ whose sections vanish on Z.

Suppose X is proper over S with connected fibers. Let

$$(\mathscr{H}^1_{DR}(X/S, Z), \nabla) = \mathscr{O}_{S^{an}} \otimes_{\mathscr{O}_S} (H^1_{DR}(X/S, Z), \nabla) .$$

We claim, for $Z \subseteq X$ finite over S.

$$(\mathscr{H}^1_{DR}(X/S, Z), \nabla) \cong (\mathscr{O}_{S^{an}} \otimes \mathscr{H}^1(X/S, Z, \mathbf{C}), d \otimes id)$$

This follows from the relative Poincaré lemma above on S' and hence on all of S since both sides are integrable connections. Hence,

LEMMA 1.6.2. *There is a natural isomorphism*

$$(\mathscr{H}^1_{DR}(X/S, Z)^\vee, \overset{\vee}{\nabla}) \cong (\mathscr{O}_{S^{an}} \otimes \mathscr{H}_1(X/S, Z, \mathbf{C}), d \otimes id) .$$

In particular

$$H^i(\mathscr{H}^1_{DR}(X/S, Z)^\vee, \overset{\vee}{\nabla}) \cong H^i(S^{an}, \mathscr{H}_1(X/S, Z, \mathbf{C})) .$$

We conclude, using this, Proposition 1.1.1 and GAGA that

THEOREM 1.6.3. *There exists a natural isomorphism*

$$\beta : \mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S) \to H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{C})) .$$

b. *End of Analytic Proof*

Now suppose X is an Abelian scheme over S. We have an exact sequence of sheaves over S^{an},

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{L}ie_{X^{an}/S^{an}} \to \underline{X}^{an} \to 0 .$$

From the corresponding long exact sequence of cohomology groups we obtain an exact sequence

$$\mathscr{L}ie_{X^{an}/S^{an}}(S^{an}) \to X^{an}(S^{an}) \xrightarrow{\delta} H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) .$$

We may describe $\delta(s)$ as follows: Suppose $e \neq s$. Let $Z = e \cup s$. Then as $f_*(\Omega^1_{X^{an}/S^{an}})$ maps into $\mathscr{H}^1_{DR}(X/S)$, $\mathscr{H}_1(X/S, Z, \mathbf{Z})$ maps into

$$f_*(\Omega^1_{X^{an}/S^{an}})^\vee = \mathscr{L}ie_{X^{an}/S^{an}}$$

so that the diagram

$$\begin{array}{ccc} \mathscr{H}_1(X/S, Z, \mathbf{Z}) & & \\ \uparrow & \searrow & \\ \mathscr{H}_1(X/S, \mathbf{Z}) & \to & \mathscr{L}ie_{X^{an}/S^{an}} \end{array}$$

commutes. Let $\mathscr{U}$ be an ordered covering of S by open disks. For each $U \in \mathscr{U}$ let $\gamma_U \in \mathscr{H}_1(X/S, Z, \mathbf{Z})(U)$ such that $\gamma_U \to 1$ under the map $\mathscr{H}_1(X/S, Z)(U) \to \mathbf{Z}$. Then the image of γ_U in $X(U)$ is $s(U)$. Hence $\delta(s)$ is represented by the one cocycle $\{(U, V) \to \gamma_U - \gamma_V\}$.

Now, it follows from this and the remark after Proposition 1.6.1 that $\beta \circ M$ is equal to the composition of δ and the natural map

$$H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) \to H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{C}) \cong H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) \otimes \mathbf{C} .$$

Hence, if $s \in X^{an}(S^{an}), M(s) = 0$ iff there exists a positive integer n such that $\delta(ns) = n\delta(s) = 0$. Hence ns is in the image of $\mathscr{L}ie_{X^{an}/S^{an}}(S^{an}) \to X^{an}(S^{an})$ and so is an infinitely divisible element of $X^{an}(S^{an})$.

Suppose $s \in X(S)$. We claim ns is an infinitely divisible element of $X(S)$. Let m be a positive integer. Let $t \in X^{an}(S^{an})$ such that $mt = ns$. There exists a finite étale Galois covering $\tilde{S}$ of S such that $t \in X(\tilde{S})$. If $\sigma \in \mathrm{Gal}(\tilde{S}/S)$, then $t^\sigma = t$ because $t^\sigma(x) = t(\sigma^{-1}(x))$ for $x \in \tilde{S}(\mathbf{C})$. It follows that $t \in X(S)$. This establishes our claim.

Finally, it follows from the function field Mordell-Weil Theorem [LN] that the image of ns in $X_{\mathbf{C}(S)}(\mathbf{C}(S))$ is a constant section X/S. Theorem 1.4.3 now follows immediately. □